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"Thermal" equilibration and the statistics of ocean currents

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ABSTRACT. Much can be learned by rattling a box of red and blue marbles. However, the statistics of ocean currents may be more complicated.

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Contents:

1. Simple ideas

Evolution of large scale flow in the oceans or atmosphere, thought of as interactions among a large but finite number of degrees of freedom, has been modelled in a variety of numerical simulations. The statistical behavior of these numerical models, though a considerable simplification of reality, remains poorly understood. Ideas from turbulence phenomenology such as similarity subranges and cascade mechanisms have only limited application over the relatively limited wavenumber ranges available to numerical models. Analytical closure theories require uncertain hypotheses and then apply only to the most idealized cases, e.g. to statistically homogeneous flow, and even then at considerable labor. The situation invites more simpleminded interpretation.

In this paper, we show how simple ideas of temperature equilibration, the bringing together of hot things and cold things to make warm things, can provide a reliable qualitative understanding of much that is seen in numerical simulations. Here we refer not to the physical temperature of seawater but rather to generalized statistical mechanical "temperatures" of the fluid flow. A curiosity is that such temperatures may be negative as well as positive. However, we deemphasize the sometimes mystique of "negative temperature flow".

Consider not the oceans but the oft-used box of red and blue marbles initially segregated red on left, blue on right. The box is subjected to a prescribed mechanical agitation. A deterministic calculation of the trajectories of marbles, even assuming rigid spheres, is implausible. A statistical calculation, perhaps in terms of diffusive fluxes, can be quite complicated. The simple thing to say is that, averaged over some sufficiently long time, each species of marble will tend to become spread uniformly about the box. Of course, for any given agitation one may choose some very special initial arrangement of marbles which will not lead to mixing. However, all such non-ergodic initial conditions are presumed to occupy zero measure in a suitably defined initial condition space. (On the other hand, the human mind displays a frightening ability to draw examples from this zero measure set.)

The box of marbles is the archetypal example of reversible dynamics leading to irreversible statistical evolution. Unhappily, the Navier-Stokes equations, because of dissipation (a statistical process at molecular scale), are already irreversible and it is this fact which, in large part, ultimately makes oceanography more difficult than the box of marbles. For the moment, we pursue the marbles.

2. Some definitions

A statistical mechanical system is defined in terms of a limited number of external parameters and a much larger number of internal parameters. External parameters are those given for the problem which the system in its evolution cannot change, e.g. the energy or volume of a contained system or the numbers of molecules of different species (non-reacting). Internal parameters are the set of generalized coordinates required to specify each distinguishable state of the system.

If, because of quantization, the states of the system are countable, then entropy S is defined as proportional to the logarithm of the number of states available for given ranges of values of the external parameters. More usefully, entropy is defined in terms of probability integrals over ensembles of like systems in a way that practically coincides with the number of states definition but doesn't depend on quantization.

Now it is interesting to ask how entropy depends upon the external parameters. The variations of S with respect to external parameters bring in the useful thermodynamic quantities: temperature, pressure, chemical potential, etc. Ordinary temperature θ is defined

$$\frac{1}{\theta} = \frac{\partial S}{\partial E}$$

where E is the total energy of the system and the partial derivative is evaluated holding all other external parameters constant. Rather than going on to abuse a whole list of thermodynamic concepts, we will generalize temperature as follows: Let the set of external parameters be $\underline{X} = \{x_i;\ i=1,\ldots,n\}$. Define a "temperature of the i-th type" θ_i as

$$\frac{1}{\theta_i} = \frac{\partial S}{\partial x_i}\bigg|_{\underline{X}-x_i}$$

where the only caution required is that the reader avoid any inclination to equilibrate temperatures of different types. Different temperature types are totally different quantities though they may be related by an equation of state, as are the ordinary temperature and pressure of a gas.

3. Inhibitions and pseudo-temperatures

The box of marbles doesn't suggest very many interesting external parameters--perhaps the chemical (color?) potentials. However, we may apply an inhibition in which we suppose that we may fix some internal parameter of the system, creating in effect another external parameter. Suppose we inhibit the value of the j-th internal parameter $y_j \in \underline{Y}$, $\underline{Y} = \{y_k;\ k=1,\ldots,m\}$. By adding y_j to the list of external parameters, we may obtain a pseudo-temperature

$$\frac{1}{\tilde{\theta}_j} = \frac{\partial S}{\partial y_j}\bigg|_{\underline{X}}$$

For the marbles, let y be the distance from the left end of the box (of unit length) to the center of mass of the red marbles. If the marbles are equally apportioned and fill the box, y has a minimum value $y=.25$, a maximum value $y=.75$ and an expected value $y=.5$. By assuming an inhibition on y (as could be accomplished by means of semipermeable membranes passing only red or blue marbles), we obtain a pseudo-temperature $\tilde{\theta}$. For $y=.25$, the filled box of marbles is in an unique configuration, $S=0$, and $\tilde{\theta}=0^{\circ}$ absolute. Allowing y to increase slightly, say by pairwise exchanges of marbles, entropy increases rapidly so that $\tilde{\theta}$ has a small positive value. Near $y=.5$, entropy becomes a stationary function of y and $\tilde{\theta}\rightarrow+\infty$. Constraining y to values greater than .5, $\tilde{\theta}$ is at first large and negative becoming small and negative until as $y\rightarrow.75$, $\tilde{\theta}\rightarrow-0^{\circ}$ absolute.

The idea of temperature may seem contrived in this simple example. But observe how quickly the idea becomes interesting if we allow gravity to act from one end of the box and suppose the red marbles to be of more dense composition than the blue marbles. Also, a laboratory example of a system much like our box of marbles, and in which negative temperature states can be constructed, is the case of a paramagnetic dielectric in an impressed magnetic field (Landau and Lifshitz, 1958).

4. Temperature and fluid flow

Onsager (1949) considers the representation of an inviscid two-dimensional fluid flow in a bounded domain in terms of interactions among a collection of isolated line vortices. If the positive and negative vortices are quite spread about and nearly paired off, their velocity fields tend to cancel and the system has some near minimum kinetic energy. (The interaction energy but not the "self energy" associated with line vortices is here considered.) If positive and negative vortices are segregated into large clumps of like signed vorticity, their velocity fields add, approaching some maximum attainable kinetic energy of the system. At either extreme energy, the possible configurations of the vortices are quite restricted. At a special value E_0 the number of configurations, or entropy, is a maximum. Now the addition of a small energy near E_{min} allows a large increase in entropy. The temperature (with respect to energy) is small and positive. Addition of more energy causes the temperature to increase until at $E=E_0$ the temperature is singular. Still more energy results in a temperature which is large and negative becoming small and negative as E approaches E_{max}.

Motion of the line vortices is identical to the motion of massless line charges oriented along an uniform magnetic field. As a model of plasma interactions, this problem has enjoyed a considerable literature which largely emphasizes the occurrence of negative temperature states.

For our purpose, it is useful to Fourier transform the vorticity field. However, this requires an infinite number of Fourier modes together with an infinite number of implicit constraints which preserve the isolated line vortex nature of the flow. Presumably the actual number of (complex) degrees of freedom among the Fourier modes is just the number of line vortices. Thus we truncate, ad hoc, the Fourier modes above some wavenumber k_{max}, retaining a number of complex Fourier coefficients equal to the number of line vortices. Although the problem is now fundamentally altered, behavior remains qualitatively much the same, with definite minimum and maximum attainble energies for given mean square vorticity ("enstrophy") and exhibiting negative temperature behavior.

5. Oceanographic applications

Large scale flow in the oceans and atmosphere is often depicted as two-dimensional or as consisting of a relatively small number of layers of two-dimensional flow. Prima facie we hope to apply some of the above ideas. The appropriate disclaimers are discussed later.

Barotropic flow over topography

Let the velocity streamfunction be $\psi(x,y,t)$, the vorticity $\zeta = \nabla^2\psi$, and let the Coriolis parameter times fractional height of topography (relative to mean depth) be $h(x,y)$. Then

$$\frac{\partial}{\partial t}\zeta + \frac{\partial(\psi,\ \zeta + h)}{\partial(x,y)} = 0$$

Integrals of the motion are

energy $\frac{1}{2}\overline{|\nabla\psi|^2}$ and "total enstrophy" $\frac{1}{2}\overline{(\zeta+h)^2}$

where overbars denote integration over the bounded domain of the flow. We inquire how statistics of the flow may relate to staistics of the topography.

Energy and total enstrophy are external parameters of the system relative to which we define first- and second-temperatures θ_1 and θ_2, respectively. For any given topography, Fourier coefficients of h are also external parameters though these will not interest us as such. Internal parameters could be the Fourier coefficients of ψ, though it is useful to recast these in terms of spectral shapes and correlations between, say, ζ and h. Finally we seek expected values of some of these internal parameters.

We hypothesize an inhibition on the overall correlation $\overline{\zeta h}$, obtaining a pseudo-external parameters $\overline{\zeta h}$ and $\overline{\zeta^2}$ since $\overline{(\zeta+h)^2}$ remains fixed. We have pseudo-temperatures $\tilde{\theta}_{\overline{\zeta h}}$ and $\tilde{\theta}_{\overline{\zeta^2}}$. Suppose we fix $\overline{\zeta h}=0$, so the flow is uncorrelated from h and hence in a maximum entropy condition with respect to $\overline{\zeta h}$. Thus $\tilde{\theta}_{\overline{\zeta h}} = \pm\infty$. If there were no restriction on $\overline{\zeta^2}$, the flow could evolve to energy equipartition as a maximum entropy state. At energy equipartition, $\overline{\zeta^2}$ has some value, say $\overline{\zeta^2}_0$, for which $\tilde{\theta}_{\overline{\zeta^2}} = \pm\infty$. In fact, geophysical spectra are far from energy equipartition, tending to have much more energy in low than in high wavenumbers. Thus it is appropriate to consider values of $\overline{\zeta^2}$ much less than $\overline{\zeta^2}_0$, so that $\tilde{\Theta}_{\overline{\zeta^2}}$ should be thought of as positive but small (cold).

Upon lifting the inhibition on $\overline{\zeta h}$, or effectively bringing the flow into thermal contact with the topography, $\tilde{\Theta}_{\overline{\zeta h}}$ and $\tilde{\Theta}_{\overline{\zeta^2}}$ begin to equilibrate. $\overline{\zeta h}$ becomes negative, allowing an increase in $\overline{\zeta^2}$. Thus $\tilde{\Theta}_{\overline{\zeta h}}$ is cooled from $+\infty$ while $\tilde{\Theta}_{\overline{\zeta^2}}$ is warmed from its initial chilly value. Equilibration is attained at the value of the external temperature Θ_2.

Actual values of Θ_1 and Θ_2 can be calculated in terms of the external parameters, the energy, the total enstrophy and the statistics of the topography, in what is effectively an equation of state. However, we gain more intuition by characterizing $\tilde{\Theta}_{\overline{\zeta h}}$ as the temperature of a reservoir, call it the "topography", and $\tilde{\Theta}_{\overline{\zeta^2}}$ as the temperature of another reservoir, the "flow". The equilibration temperature then depends upon the relative heat capacities of the flow and of the topography. Heat capacity is here an imprecise idea which must really depend upon all of the external parameters. However, we can loosely characterize the heat capacity of the topography by the overall topographic variance $\overline{h^2}$ and the heat capacity of the flow by the kinetic enstrophy $\overline{\zeta^2}_0$ which would occur if the flow could go to energy equipartition.

In summary, the evolution of barotropic flow over topography from a random initial condition is seen as the bringing into thermal contact of a infinitely hot topographic reservoir of finite heat capacity and a quite cold flow reservoir of different heat capacity to produce a warm (or cool) mix of flow+topography.

Equilibrium spectrum of linear topographic waves

A particularly interesting case results when we simply neglect the heat capacity of the flow. Then, with no change in $\overline{\zeta h}=0$ the infinitely hot topography just brings the flow to infinite temperature, i.e. to energy equipartition. A paradox appears if we note that for any given realization of h(x,y) the linear equation of motion

$$\frac{\partial}{\partial t}\zeta + \frac{\partial(\psi, h)}{\partial(x, y)} = 0$$

subjected to finite wavenumber truncation, admits some normal mode representation

$$\zeta(x,y,t) = \sum a_i \hat{\zeta}_i(x,y,t)$$

Thus we have found the time-averaged wavenumber energy spectrum of almot every normal mode $\hat{\zeta}_i$, namely energy equipartition. If this is not so, then we must say that the linear topographic wave equation, finitely truncated, is non-ergodic. However, discarding the usual zero-measure sets of special topographies and special initial flows, it is hard to see why the evolution should not be ergodic.

Flow on a beta plane

Consider rigid lid, inviscid, barotropic flow on a laterally bounded β-plane, i.e.

$$\frac{\partial}{\partial t}\zeta + \frac{\partial(\psi, \zeta+\beta y)}{\partial(x,y)} = 0$$

spectrally truncated. External parameters are energy $\frac{1}{2}\overline{|\nabla\psi|^2}$ and total enstrophy $\frac{1}{2}\overline{(\zeta+\beta y)^2}$.

Some useful internal parameters are the correlation $\overline{\zeta y}$ and the energy spectrum. Again let us focus on the correlation, assuming an inhibition on $\overline{\zeta y}$ and so creating pseudo-temperatures $\tilde{\Theta}_{\overline{\zeta y}}$ and $\tilde{\Theta}_{\overline{\zeta^2}}$. Pick $\overline{\zeta y}=0$, hence $\tilde{\Theta}_{\overline{\zeta y}} = \pm\infty$. The appropriate choice of $\tilde{\Theta}_{\overline{\zeta^2}}$ is, again, that it be small and positive.

Upon lifting the inhibition, we bring the "flow" into thermal contact with the "basin", the latter having a heat capacity dependent on β^2. A cold flow in an infinitely hot basin evolves to a warm or cool "flow+basin", here characterized by $\overline{\zeta y} < 0$, i.e. a tendency to sort negative (CW) vorticity into the northern basin and positive (CCW) vorticity into the southern basin. If the net circulation $\overline{\zeta}$ is zero, the equilibrium flow has a steady mean component: a broad westward drift in the interior with swift eastward returning currents along the north and south boundaries. Nonzero choices of $\overline{\zeta}$ may shift the eastward current more to either the north or south boundary.

In this problem, a most interesting feature is the mechanism of the approach to equilibrium. The decrease in $\overline{\zeta y}$ with increase in $\overline{\zeta^2}$, which is the cooling of the basin and the warming of the flow, is accomplished by intensification of currents along the western boundary. Western intensification here follows from the Second Law of Thermodynamics as surely as that the red and blue marbles, initially segregated, will tend to mix. Of course, in equilibrium the marbles will be mixed and the western intensification will vanish.

Two-layer flow

In a first order representation of the vertical structure of ocean flow, one may consider coupled equations for the evolution of two immiscible fluids contained between two rigid flat surfaces, without β, viz.

$$\frac{\partial}{\partial t} Q_1 + \frac{\partial(\psi_1, Q_1)}{\partial(x,y)} = 0$$

$$\frac{\partial}{\partial t} Q_2 + \frac{\partial(\psi_2, Q_2)}{\partial(x,y)} = 0$$

where $Q_1 = \zeta_1 + F_1(\psi_2 - \psi_1)$, $Q_2 = \zeta_2 + F_2(\psi_1 - \psi_2)$, $\zeta_i = \nabla^2\psi_i$ and F_1 and F_2 are coupling coefficients depending on the thicknesses of each layer and the density difference between the two fluids. Subscripts are layer indices. F_1 and F_2 are nondimensionalized by a length scale, the internal radius of deformation, which is given in dimensionfull form by $R = 1/\sqrt{F_1 + F_2}$

External parameters are the sum kinetic energies plus the potential energy of the interface

$$\bar{E} = \frac{1}{F_1}\overline{|\nabla\psi_1|^2} + \frac{1}{F_2}\overline{|\nabla\psi_2|^2} + \overline{(\psi_1 - \psi_2)^2}$$

and the total enstrophy in each layer, separately, $\overline{Q_1^2}$ and $\overline{Q_2^2}$. Relative to these external parameters, we have three external temperatures θ_1, θ_2 and θ_3 respectively.

Once again our attention is drawn to the correlation between layers, $\overline{\zeta_1\zeta_2}$, on which we will assume an inhibition. However, the problem has become much more complicated because changes in $\overline{\zeta_1\zeta_2}$ will cause simultaneous changes among the terms appearing in $\overline{E}$, $\overline{Q_1^2}$ and $\overline{Q_2^2}$.

Suppose we fix $\overline{\zeta_1\zeta_2}=0$ creating a reservoir, the "interface", distinct from another reservoir, the "flow". Lifting this inhibition brings the flow into contact with the interface. However, we must think of this contact in either of two ways: we may say that the reservoirs exchange three different species of things until three different kinds of pseudo-temperatures equilibrate; or we may assume a simple thermal contact but with some difficulty determining what "heat" is exchanged. The latter interpretation, though awkward, turns out to be the easier one and follows more nearly our earlier examples.

$\overline{\zeta_1\zeta_2}=0$ puts the temperature of the interface at infinity. But what is the temperature of the flow in terms of the "heat" that the interface may provide? The problem is that the interface is a source simultaneously of kinetic energy as well as kinetic enstrophy in each layer. This brings us to the question how, in each layer, does entropy vary with kinetic energy? The answer is that, if there is no constraint on kinetic energy, a flow will evolve to enstrophy equipartition which therefore corresponds to $\Theta_1=\pm\infty$. Typical geophysical spectra are more "red" than enstrophy equipartition implying that any increase in kinetic energy will only further restrict the flow. Hence $\Theta_1 < 0$. Thus entropy change in the flow depends on the competing effects of negative Θ_1 and positive Θ_2 and Θ_3.

The tradeoff between increasing entropy with increase in kinetic enstrophy and decreasing entropy with increase in kinetic energy is dependent on the length scale under consideration. Near a wavenumber k, changes in $\overline{\zeta_1\zeta_2}$ cause changes in kinetic enstrophy and kinetic energy in approximately the ratio

$$k^2\left(1+{}^1\!/\!_{k^2R^2}\right)$$

For $k^2R^2 \gg 1$, the ratio is just k^2 which corresponds to increasing excitation in a spectrum which merely retains its shape. However, entropy is more dependent on the shape than amplitude of the spectrum and therefore remains nearly stationary to exchanges with the interface. Thus for short length scales, we can consider a single temperature of the flow which is positive at any length scale and which is quite hot.

For $k^2R^2 \ll 1$, the ratio of enstrophy to energy production goes to $1/R^2$ and so does not decrease as k^2 gets small. Thus a positive correlation $\overline{\zeta_1\zeta_2}$ in larger length scales acts preferentially as a source of kinetic enstrophy, allowing the spectrum to shift to higher wavenumbers thereby generating substantial entropy. Large scale flow acts as a reservoir at positive but low temperature. Clearly the contrivance is that we are assigning different temperatures to different portions of the spectrum in lieu of considering simultaneous transfer of different properties.

At last the picture becomes simple. Fixing $\overline{\zeta_1\zeta_2}=0$ creates an interface at infinite temperature and a flow which, relative to changes in $\overline{\zeta_1\zeta_2}$, is quite hot for length scales shorter than the deformation radius R but which is much colder in scales larger than R. In thermal contact, the interface is cooled only slightly at short length scales but is chilled substantially in large scales. Thus in scales larger than R, motion in the two layers locks together becoming barotropic. Shorter length scales remain poorly correlated, a mixture of barotropic and baroclinic modes.

6. A plasma application

As noted earlier, the equations of inviscid two-dimensional flow also describe the two-dimensional motion of a plasma in an uniform normal magnetic field. If the charge density is $\rho(x,y,t) = -\nabla^2\phi(x,y,t)$ where ϕ is the electric potential, the motion is

$$\frac{\partial}{\partial t}\nabla^2\phi + \frac{\partial(\phi, \nabla^2\phi)}{\partial(x,y)} = 0$$

Consider this motion in a field of bound (immobile) charges of density $\sigma = -\nabla^2\chi$. Now the motion is

$$\frac{\partial}{\partial t}\nabla^2\phi + \frac{\partial(\phi+\chi,\nabla^2\phi)}{\partial(x,y)} = 0$$

with integrals of the motion:

energy $\frac{1}{2}\overline{|\underline{\nabla}\phi|^2} + \overline{\nabla\phi\cdot\nabla\chi}$ and charge variance $\overline{(\nabla^2\phi)^2}$.

By placing an inhibition on $\overline{\rho\sigma}$ we create a reservoir, the "bound charge distribution" and another reservoir, the "free charge motion". The temperature of the bound charge reservoir at $\overline{\rho\sigma}=0$ is infinite. The temperature of the free charge motion is given by the change of entropy with respect to electric field variance $\overline{|\underline{\nabla}\phi|^2}$. Because the electric field variance is the energy of this system, we are using "temperature" in its ordinary meaning.

The singular temperature of the plasma occurs when the spectrum of the electric field variance corresponds to equipartition of charge variance. An electric field which is less energetic than charge equipartition has a positive temperature while a more energetic field goes to negative temperature. It is the occurrence of these negative temperature states which has attracted the interest of plasma physicists. In fact though it appears that the physics of negative temperature plasmas is a straightforward extension of positive temperature physics.

At positive temperature, the presence of a positive charge at a point $(x,y)_0$ produces an enhanced probability of finding negative charges near $(x,y)_0$. At infinite temperature this probability becomes uniform over all space while for negative temperature it becomes more probable that positive charges will be found near $(x,y)_0$. If this "shielding" and "anti-shielding" effect is described in terms of a Debye length, one must say that the Debye length is finite at positive temperature becoming infinite then imaginary at negative temperature. But Debye length is a derived idea rather than a description of the motion.

Another negative temperature demon is macroscopic variability. At positive and cold temperatures, the energy spectrum is dominated by variance at short length scales which could be called microscopic or thermal fluctuations. As temperature increases, these thermal fluctuations grow in size until, at singular temperature, the largest length scales of the bounded motion dominate the energy spectrum. Going on to negative temperature, still greater fractions of the total energy condense into the largest scales of motion. Although this certainly constitutes macroscopic variability, it proceeds directly from tendencies observed in the positive temperature state with no abrupt onset in passing to negative temperature.

Now we consider the free charge motion, at some positive or negative temperature, to be brought into thermal contact with the bound charge distribution. If the free charge motion is initially at negative temperature, it is cooled toward the singular temperature while the correlation $\overline{\rho\sigma}$ becomes positive, warming the bound charge reservoir to negative temperature. If the free charge motion is initially at positive temperature, it is warmed while the bound charge reservoir is cooled to some $\overline{\rho\sigma} < 0$. For $\theta_1 > 0$, free charges tend to shield the bound charges. For $\theta_1 < 0$, free charges tend to antishield the bound charges.

7. Thermometers

An entertaining use either for topography under ocean currents or for bound charges in a plasma is their possible roles as thermometers. In its colloquial meaning, a thermometer is a device of small heat capacity and having as an internal parameter some readily observable quantity like the length of a mercury column.

A barotropic flow over a flat bottom has temperatures θ_1 and θ_2 given by the energy and enstrophy of the motion. If we suppose the problem with an irregular topography of quite small amplitude, the effect on θ_1 and θ_2 will be quadratic in the amplitude of the topography. However, the resulting normalized correlation coefficient $\overline{\zeta h}/\sqrt{\overline{\zeta^2}\,\overline{h^2}}$ is proportional to the amplitude of the topography and inversely proportional to θ_2. Thus we may insert a thermometer (the topography) which has an arbitrarily small effect on the temperatures of the flow but from which we may readily read off the second temperature.

These same comments apply to the introduction of a very weak bound charge distribution into a plasma, except that we then read the first temperature (the ordinary temperature) of the plasma.

8. Observations

We have talked mostly about correlation---with topography, with latitude, between layers, etc. The discussion can be extended to questions of spectral shape but only at considerable labor and even greater abuse of concepts. Instead we ask whether this whole construction should have any relation to reality.

In extensive numerical simulations of oceanic flow, including direct dissipation to produce more realistic dynamics, Rhines (in prep.) has observed and discussed a number of tendencies which are quite like the qualitative behavior we have discussed. For example, random flow in the presence of irregular topography is observed to relax quite rapidly into a streamfunction pattern which visibly resembles the topographic map. We also have observed this result under various kinds of dissipation (Holloway and Hendershott, 1974). The same event is observed again with still another kind of dissipation by Bretherton and Haidvogel (in prep.).

Rhines further discusses flow in two layers, initially either uncorrelated or baroclinic at large scales. The flow evolves rapidly by a kind of baroclinic instability, producing eddies at near the internal deformation radius. These eddies lock together and grow in size until the larger scale flow becomes strongly barotropic. Evolution is strikingly like our previous discussion which might be termed a "thermodynamic" account of baroclinic instability.

The real world also appears to tend to barotropic flow in scales larger than the deformation radius. Another observation from reality is that ocean currents are more swift at the western margins of ocean basins than at the eastern margins. We have seen that without wind stress or dissipation of any kind, random ocean currents will become western intensified in an effort to move toward thermal equilibration with the ocean basin. Nonetheless we are mindful that there are other theories of western intensification.

Is our qualitative agreement with observations only fortuitous? Or have we somehow manipulated these verbal arguments around to achieve known results? The answer to the latter question is no. All of the results given qualitatively above have been obtained (Salmon *et al.*, 1975) as asymptotic expectation values of the time-averaged evolution of the inviscid spectrally truncated equations of motion.

The question of fortuitous agreement remains. In particular, we recognize that inviscid truncated equations cannot, in any consistent way, be taken as an approximation to realistic flow. Dissipation, on whatever length scales it may occur, is always and essentially a part of the statistics of nonlinear flow. Here "dissipation" need not necessarily mean molecular viscosity though in principle one ultimately has viscosity (or infrared radiation loss) in mind. However, we are discussing a quasigeostrophic model of oceanic flow, and so our interpretation of dissipation is really a kind of parameterization of the failure of the model assumptions. Also, although there may be scales of motion to which dissipation, real or parametric, is indeed negligible, these nondissipative scales are coupled to other scales and ultimately to dissipative scales.

In all, it seems surprizing that we appear to find agreement with reality. More strikingly we find agreement with a variety of numerical models employing a variety of different dissipative mechanisms. How a tendency toward thermal equilibration can survive the demands of dissipation is the subject of our concluding section.

9. Caveats, disclaimers and excuses

The preceding discussion has passed by a number of fine points. The admixture of statistical mechanical concepts is imprecise or even inconsistent, e.g. as the assigning of temperature to an isolated system. Although each of these errors can be set right by more careful treatment, this kind of precise discussion of the inviscid truncated problem is just what will not carry over to the real dissipative problem. It is valuable to know how to do the inviscid truncated problem precisely, as a check against misapplication of ideas. However, the object of this paper is to develop an intuition so simple and so sturdy, albeit imprecise, that it may survive the carry over to real flow.

Ideas of dissipation and forcing can be introduced into the box of marbles, for example by replacing the righthand end of the box with a semipermeable membrane allowing red marbles sometimes to escape. The supply of red marbles is made up elsewhere, e.g. by injection through the left end or random injection over the interior. Now the stationary state departs from an uniform mix of marbles. How much is the departure? A precise answer isn't easy. A rough answer is given by comparing the residence time (average number of red marbles divided by injection rate) and the equilibration time (a characteristic time for relaxation of nonuniformities in a conservative box). When residence time is much longer than equilibration time, the mixture comes near to equilibrium. When residence time is shorter than equilibration time, substantial departure from equilibrium is expected in the stationary state.

Even in cases of relatively short residence times, some internal parameters may come nearly to their equilibrium values while others remain far from equilibrium. Although the longitudinal (left-right) gradient of red marble density may be significantly non-zero, the transverse gradient stays nearly at the equilibrium value, i.e. vanishing gradient, despite the overall state of disequilibrium.

One can go to some detail at making the non-conserving box of marbles mimic fluid turbulence, e.g. setting up cascade subranges. The serious difficulty however would arise in attempting to quantify the residence and equilibration times. Very roughly though, we may think of fluid flows in either two or three dimensions as critically damped, i.e. as having residence times (of enstrophy in 2D, energy in 3D) the order of, or somewhat less than, equilibration times. Apparently this results from dissipation or failure of the model equations acting ever increasingly effectively at higher wavenumbers. Processes which would bring about equilibration among different regions of the wavenumber spectrum ultimately cannot compete with dissipation. This failure at high wavenumbers in turn places a demand for transfer across the wavenumber spectrum until the flow is brought overall nearly to critical damping.

We expect the spectral shape to depart substantially from equilibrium as excitation cascades across scales of motion in analogy to the longitudinal gradient of red marbles. On the other hand, correlations with topography or between layers of flow may persist in the presence of a cascade, like the zero transverse gradient of marbles. However, for the fluid flow, existence of transfer depends upon maintaining triple correlations across scales of motion. This is expected to compete with, and somewhat to relax, equilibrium correlations.

Dissipation and forcing also act to sustain in the stationary state certain processes which otherwise would vanish in the approach to thermal equilibration. Intensification of currents at a western margin becomes a permanent feature of the dissipative flow, analogously to the rightward flux of red marbles. A large scale source of potential energy maintains ongoing baroclinic conversion as the "flow" attempts to thermalize with the "interface".

At a more fundamental level, the appearance of viscosity in the Navier-Stokes equations represents a bringing into contact of the flow with yet another reservoir, the field of molecular agitation. By any commensuration, changes in entropy associated with flow variables are insignificant compared to increases in entropy due to heating the molecular field. Although we have characterized the flow as hot or cold, by the same measure, the molecular field is a very nearly infinite reservoir at very nearly 0° absolute. Dissipation in the two dimensional model equations is parametric and presumed to act most effectively on higher wavenumbers, thereby dissipating enstrophy more strongly than energy. (Note that Ekman surface drag acts equally effectively on all scales. However, this form of dissipation does not obviate the need for a special high wavenumber form that must be provided in addition.)

Schematically we identify dissipation with an infinite reservoir at the absolute zero of the second temperature. The effect is to cool the second temperature of the flow towards positive 0° absolute. The equation of state simulatneously moves the first temperature toward negative 0° absolute. Interestingly, it is this state of double absolute zero which Bretherton and Haidvogel (in prep.) recommend, though in quite different words, as a model of flow over topography.

Just as dissipation is seen as a cooling of the flow, forcing mechanisms are seen as sources of warming, providing enstrophy and so bringing the second (and third) temperature up from near 0° absolute. Forcing also injects energy, warming the first temperature ever nearer to negative 0° absolute until bounded perhaps by Ekman drag. Heating with respect to second temperature (second heat?) is injected mainly to the interface, topography and basin reservoirs whereas dissipative cooling acts on the flow. This results in flow second temperatures far colder than equilibrium in thermal exchange with interface, topography and basin temperatures usually much hotter than equilibrium.

Summing up, the statistics of ocean currents are divided among a number of reservoirs identified nominally with the topography, the interface between two layers, the ocean basin and, what's left, the "flow". In isolation the reservoirs are at different temperatures. Nonlinear interaction brings the reservoirs into thermal contact, allowing the system to evolve toward equilibration. An inviscid, spectrally truncated model system achieves thermal equilibrium. Real flow however must involve forcing and dissipation mechanisms. These appear as sources of heating and cooling, applied differently to the different reservoirs, which drive the system away from equilibrium. Still, the real flow retains considerable internal freedom which is resolved by entropy maximization. The resulting tendency toward thermal equilibration is quite evident in numerical simulations and also suggests some observed real processes.

Lastly, we remark again that this discussion is only qualitative. This particular format is probably inappropriate to quantification. On the other hand, results of numerical simulations can be, and often are, taken as ends in themselves. Also we expect that quantitative theories will be developed in terms of analytical closure hypotheses. In any case though, the likely complexity of such developments will still call for some manner of simple interpretation, which is the object of this paper.

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