Dynamical Systems Analysis of Fully 3D Ocean Features

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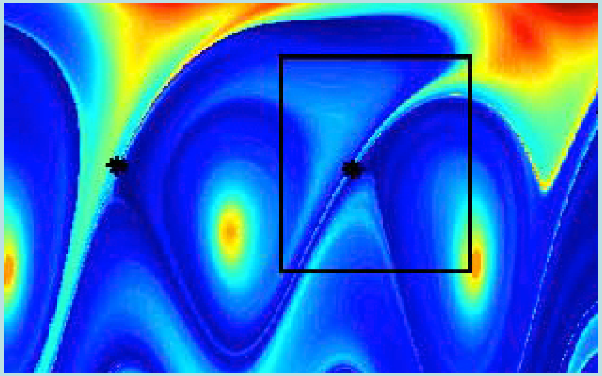
MURI: Ocean 3D+1

ONR
Office of Naval Research

WOODS HOLE OCEANOGRAPHIC INSTITUTION
1930

Chaotic Advection



•Stirring is controlled by long-lived coherent structures.

•Chaotic parcel trajectories (rapid separation in time).

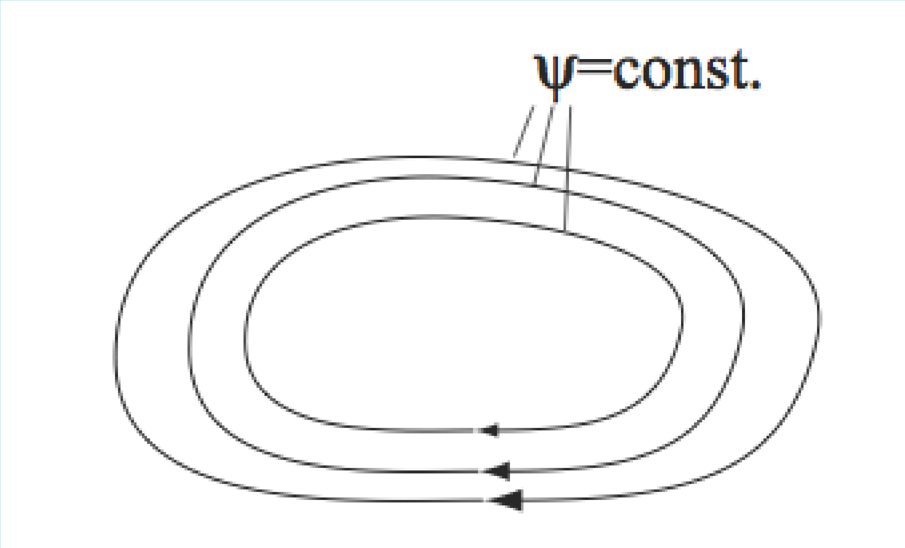
•Not all trajectories are chaotic. Barriers separate chaotic and regular regions.

•Extensive theory for flows that are 2D + time.

•Nearly all ocean examples are 2D+time or quasi 2D+time: weak vertical motion.

Do barriers and non-chaotic regions survive in flows that are fully 3D (have significant vertical motion)?

Barriers are predicted by the KAM theorem.



Every trajectory that lives on a closed streamline is time periodic, with period $T(\psi)$.

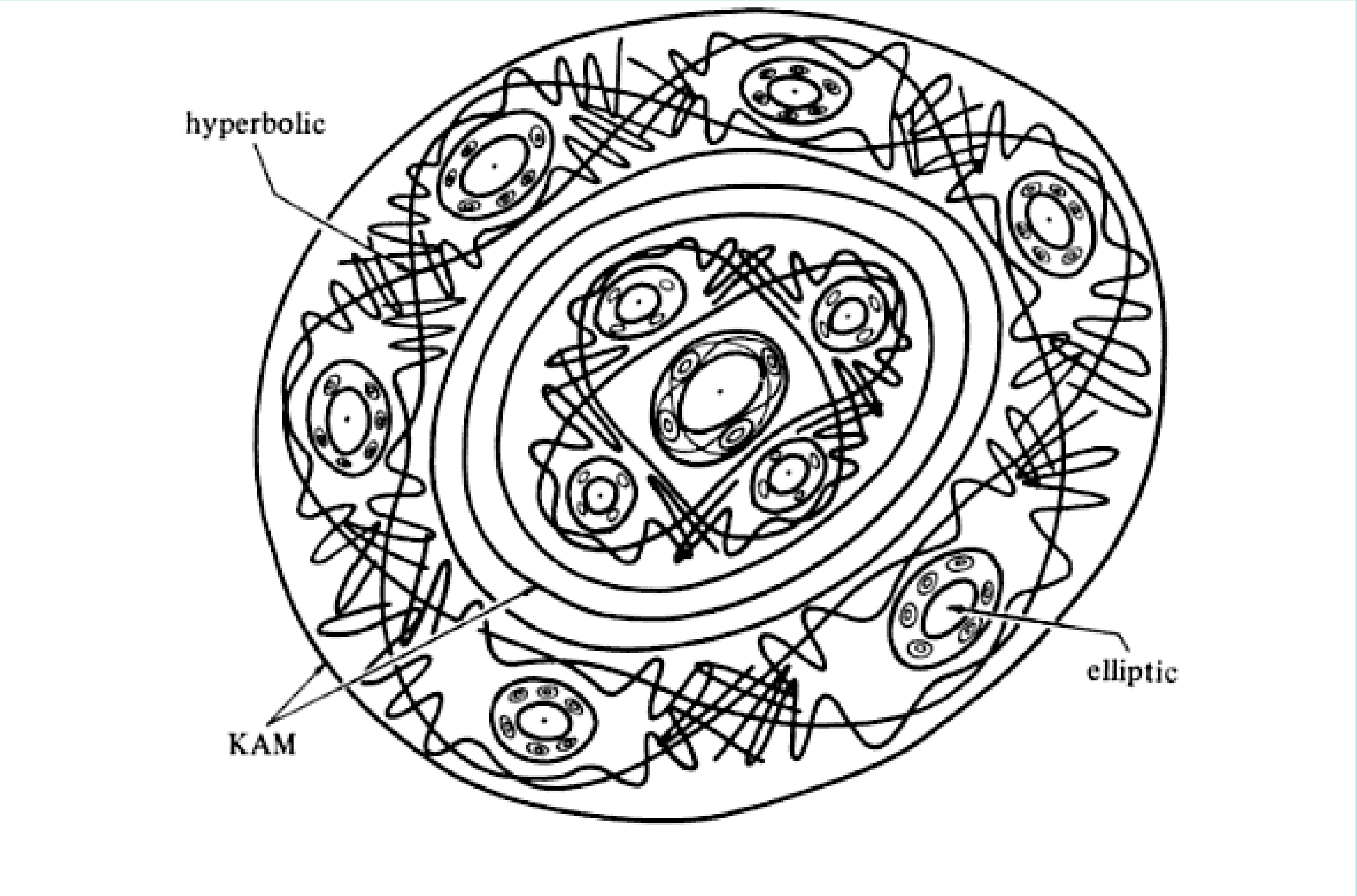
If the system is forced at period T_f, trajectories on the 'resonant' streamlines

$$T(\psi)/T_f = m/n$$

will become chaotic.

Curves with 'sufficiently' irrational $T(\psi)/T_f$ will survive. The trajectories will remain quasi periodic.

The breakup of resonant contours.



(Ottino 1989)

MURI objective: To use dynamical systems methodology to learn something about time-dependent, 3D motions in the ocean.

What is 3D? $\frac{\partial w}{\partial z}$ is significant in $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \boxed{\frac{\partial w}{\partial z}} = 0$

Main Issue: Do coherent structures exist in 3D?
(Energy cascade may be towards small scales.)

‘Rotating Can’ Experiment

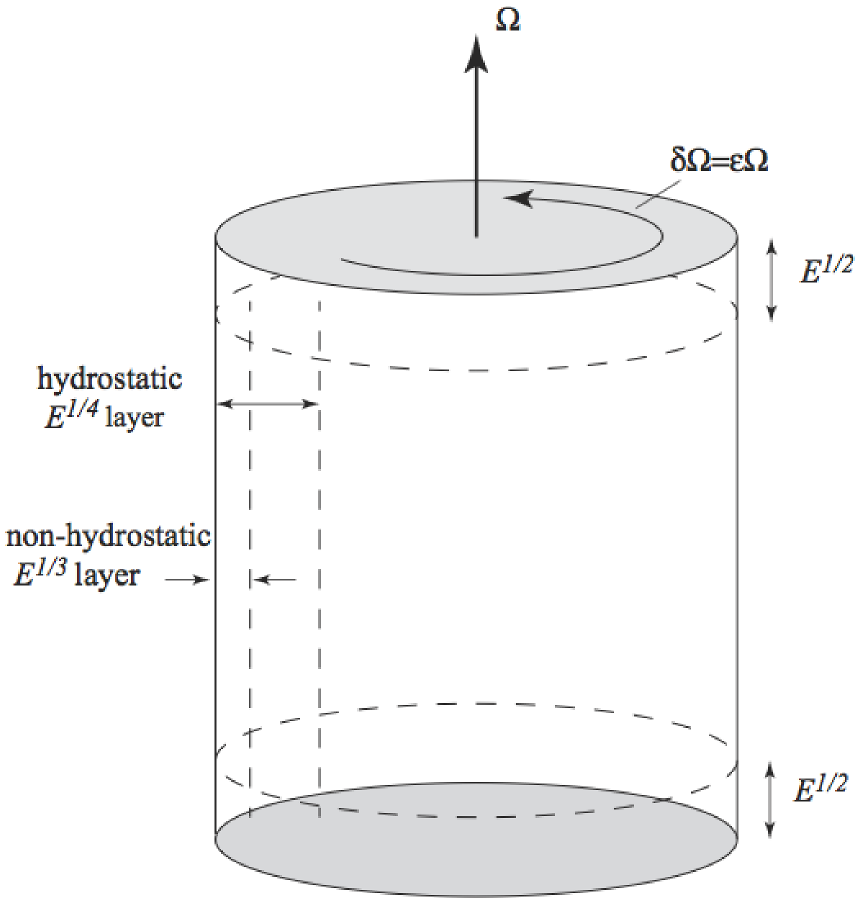
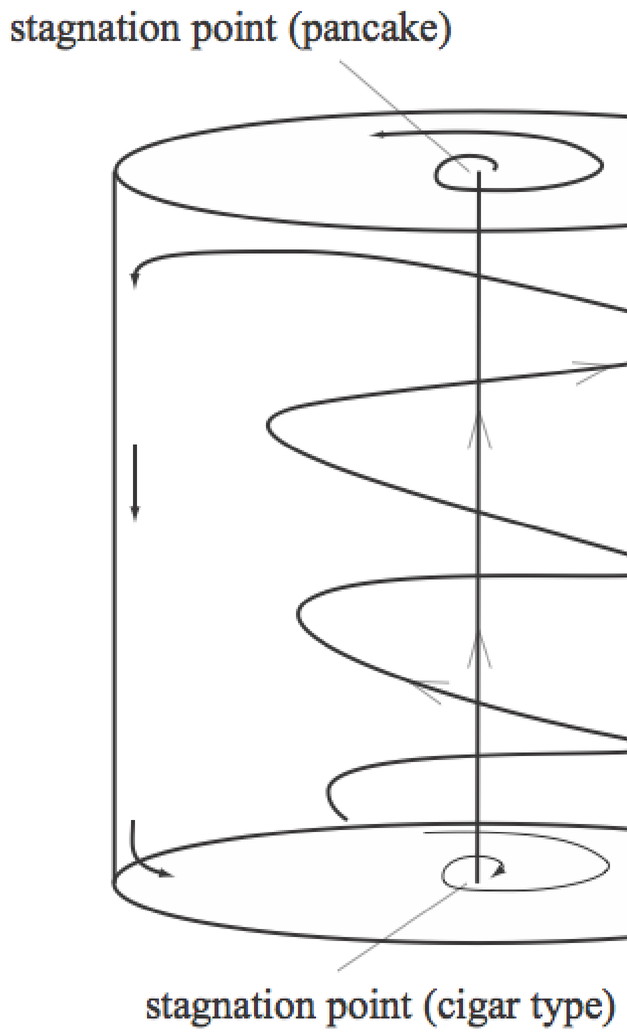


Figure 1. Rotating can with no-slip boundaries and small Ekman number. The flow is driven by a differentially rotating lid.

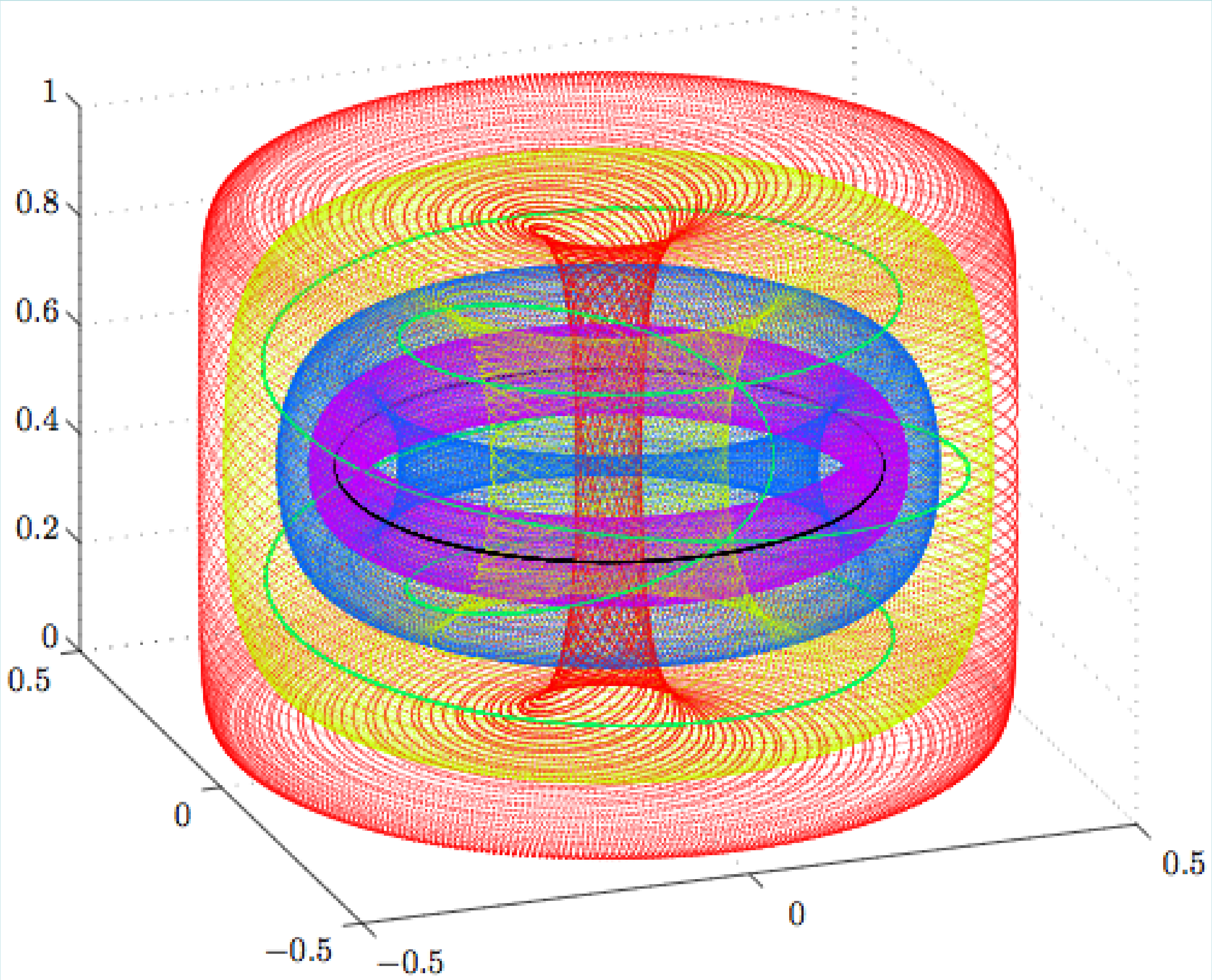

stagnation point (pancake)
stagnation point (cigar type)

Velocity Fields

1) Kinematic (3d velocity non-divergent but no dynamics)

2) Linear asymptotic solution with $E = \frac{\nu}{\Omega H^2} << 1$

3) Nonlinear numerical model.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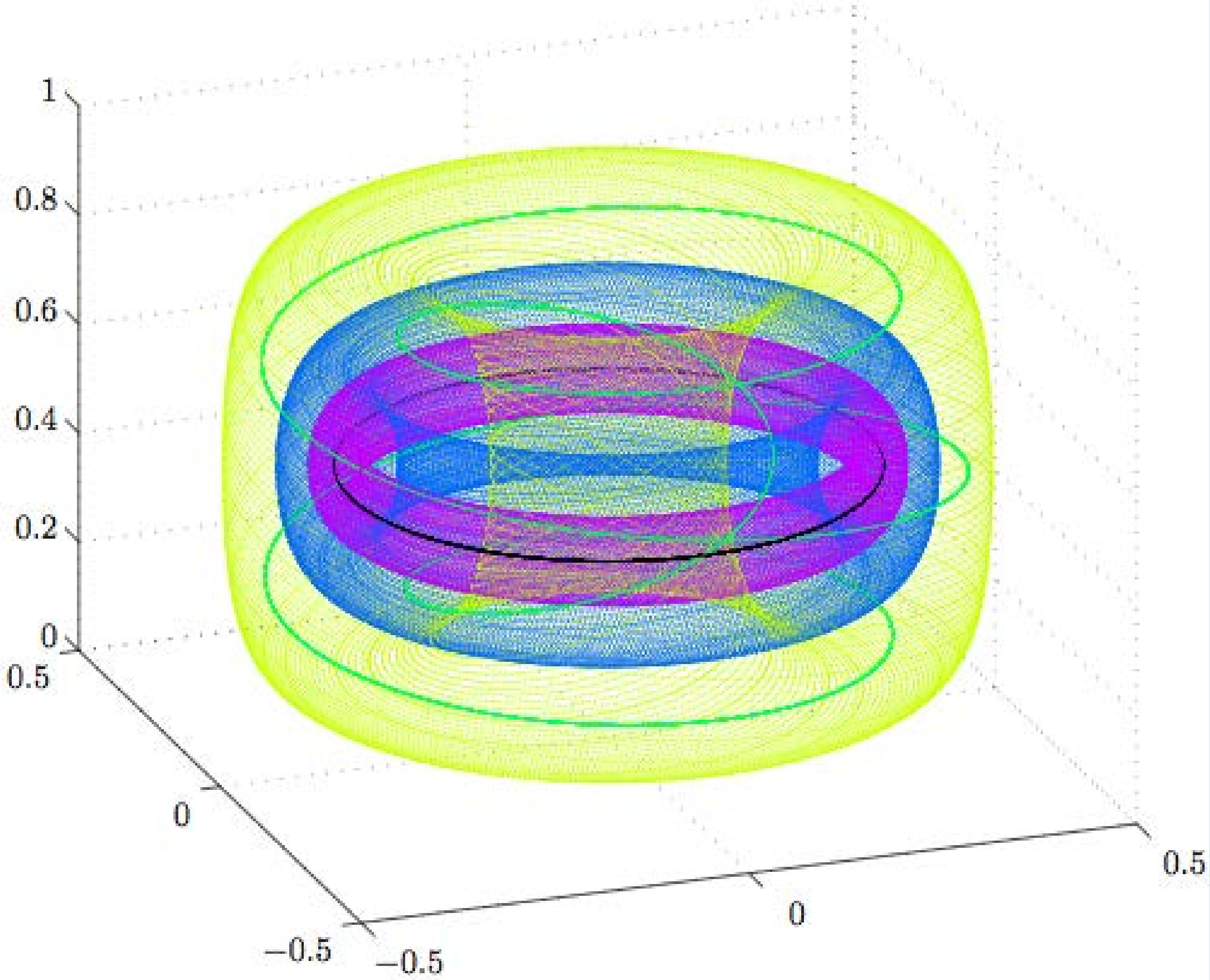
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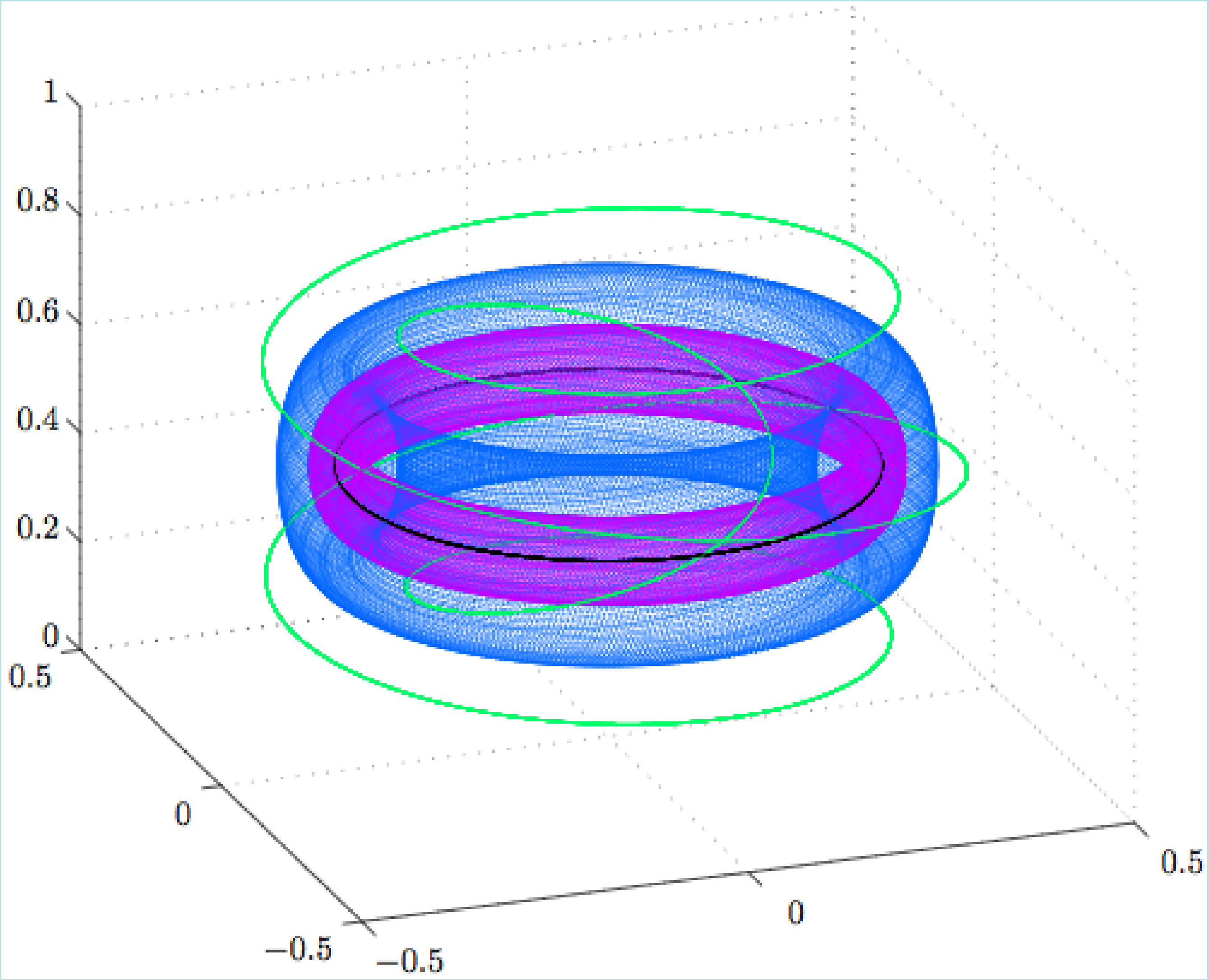
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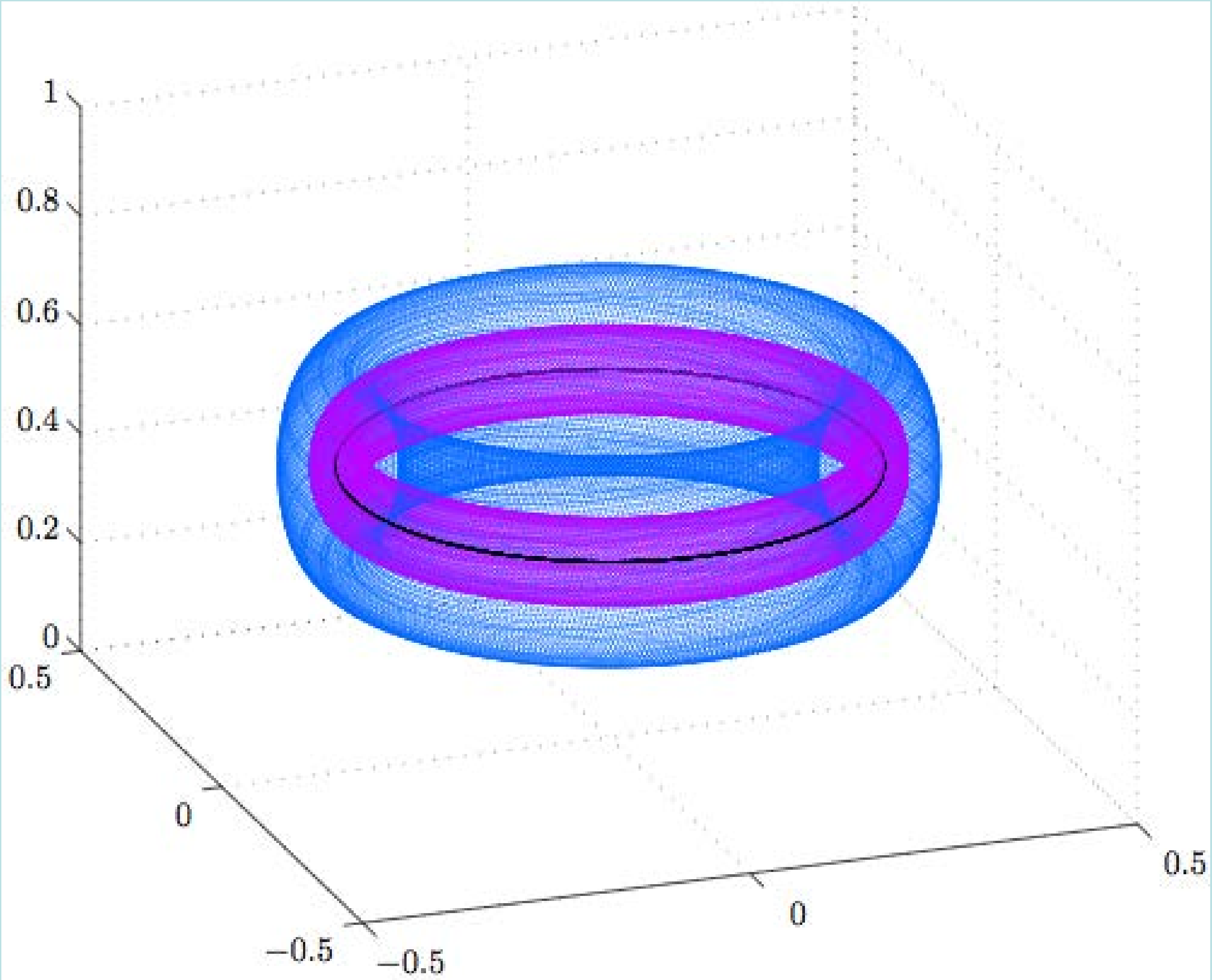

1
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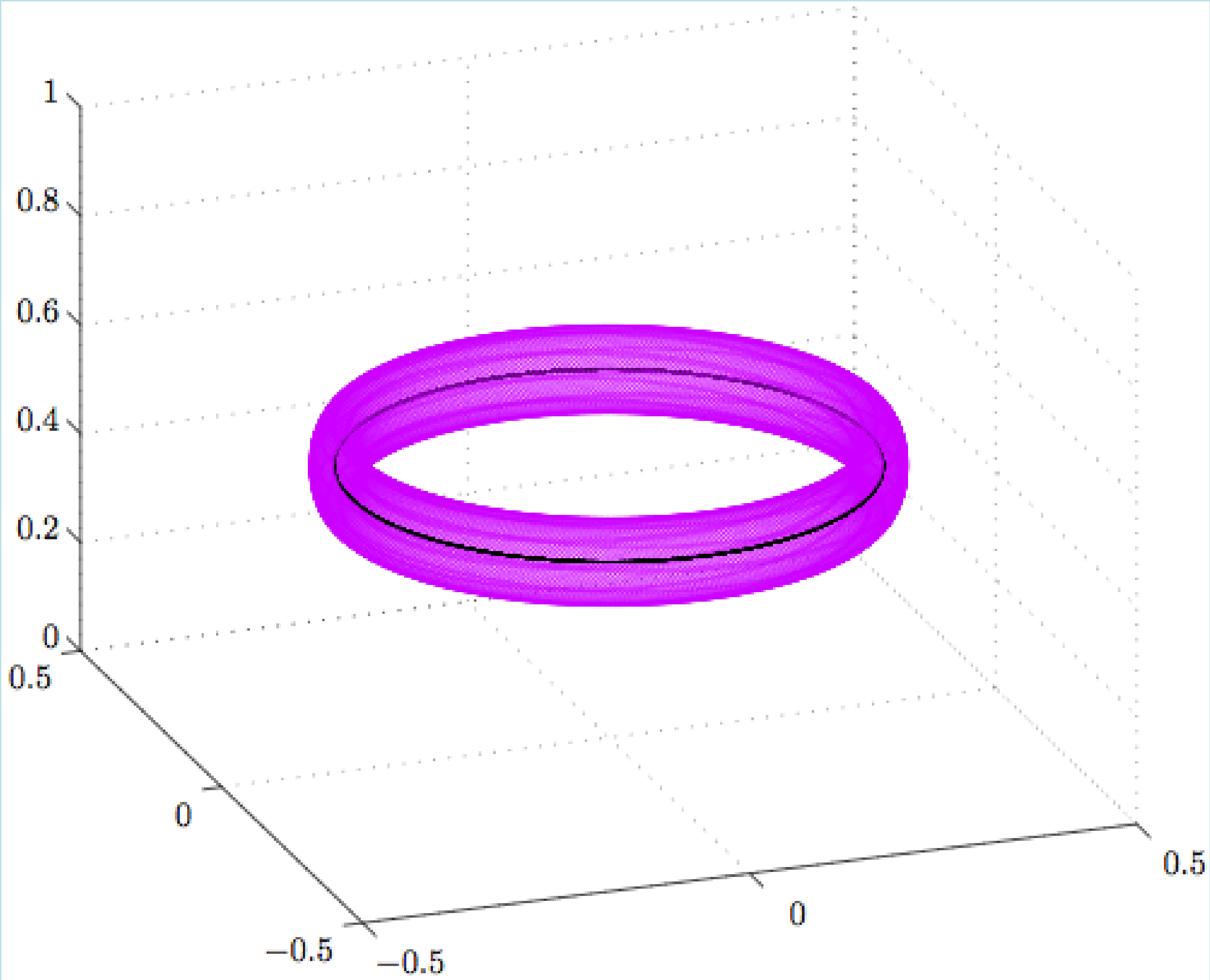
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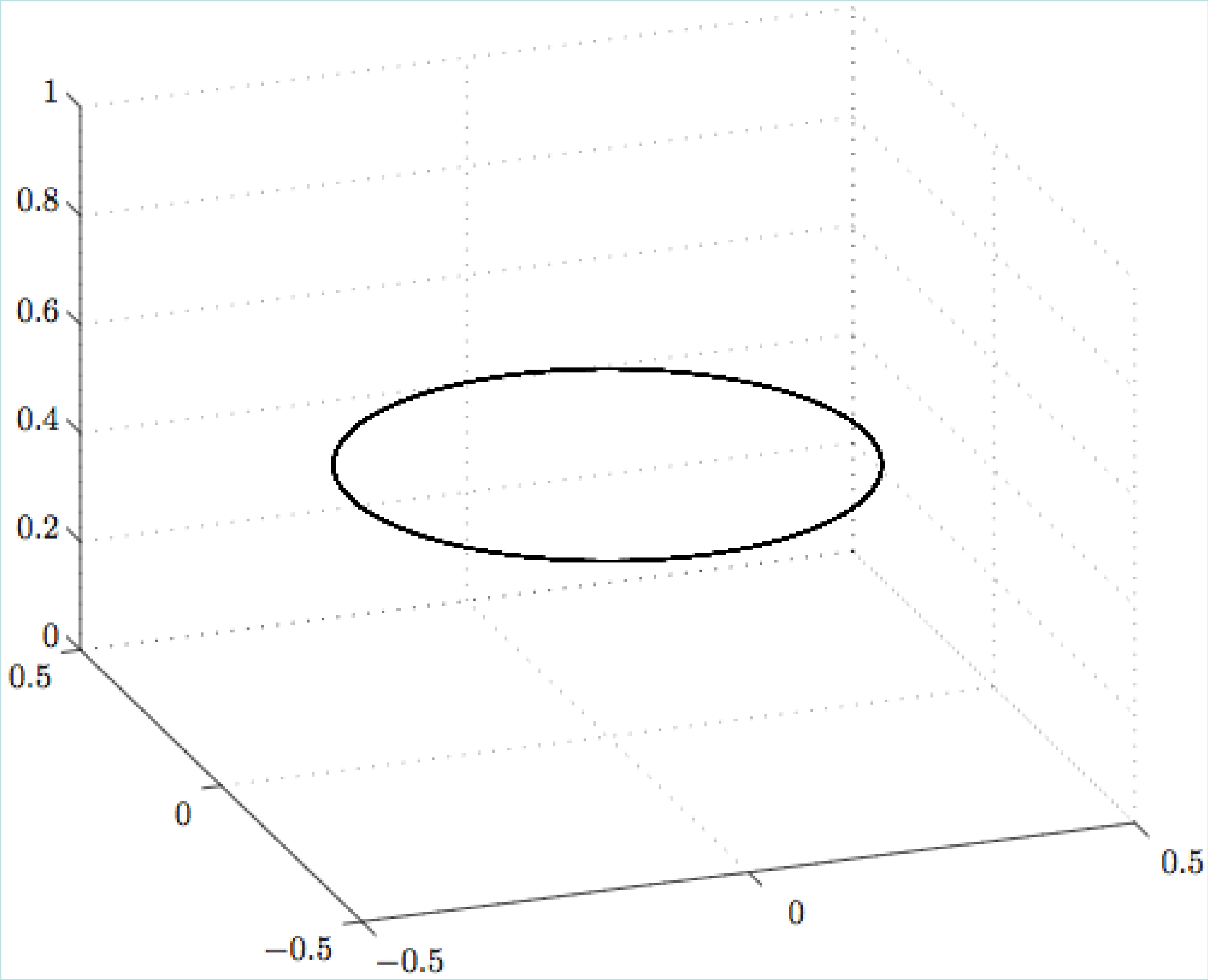
0.5



1
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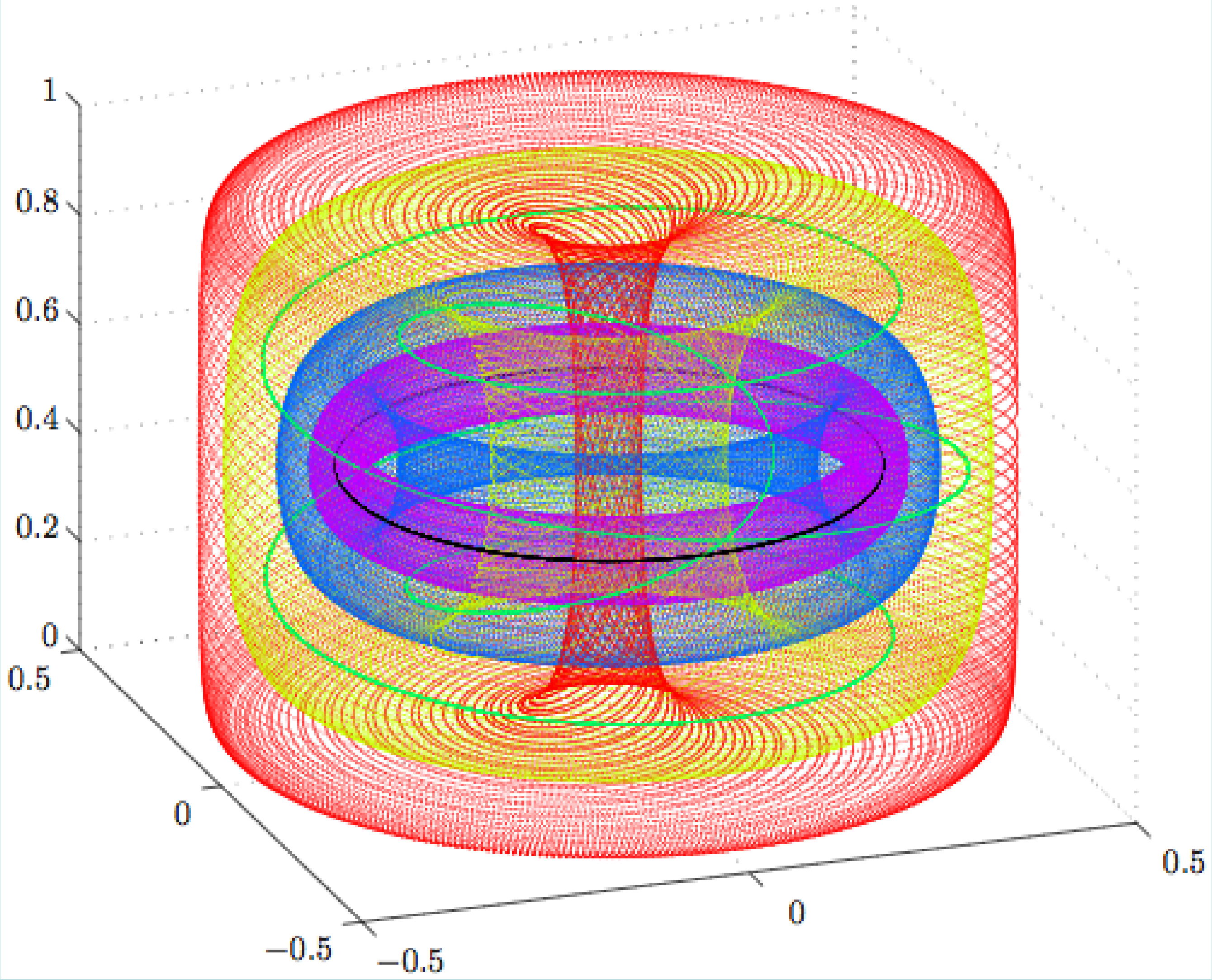
0.4
0.2
0
0.5
0
−0.5
−0.5
0
0.5


1
0.8
0.6
0.4
0.2
0
0.5
0
−0.5
−0.5
0
0.5

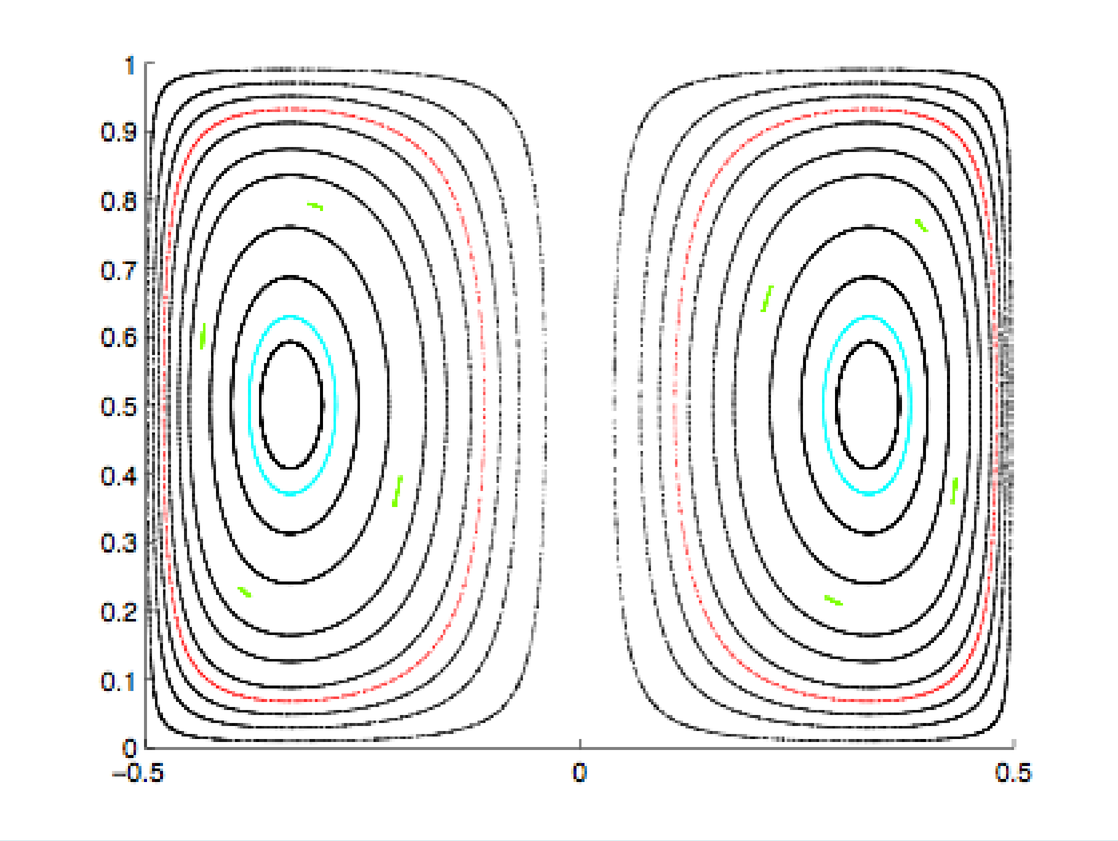
Steady Case with no θ-depedence

$$\frac{dr}{dt} = \frac{1}{r}\frac{\partial \psi(r,z)}{\partial z}$$

$$\frac{dz}{dt} = -\frac{1}{r}\frac{\partial \psi(r,z)}{\partial r}$$

$$\frac{d\theta}{dt} = \frac{V(r,z)}{r}$$

(Quasi-Hamiltonian)

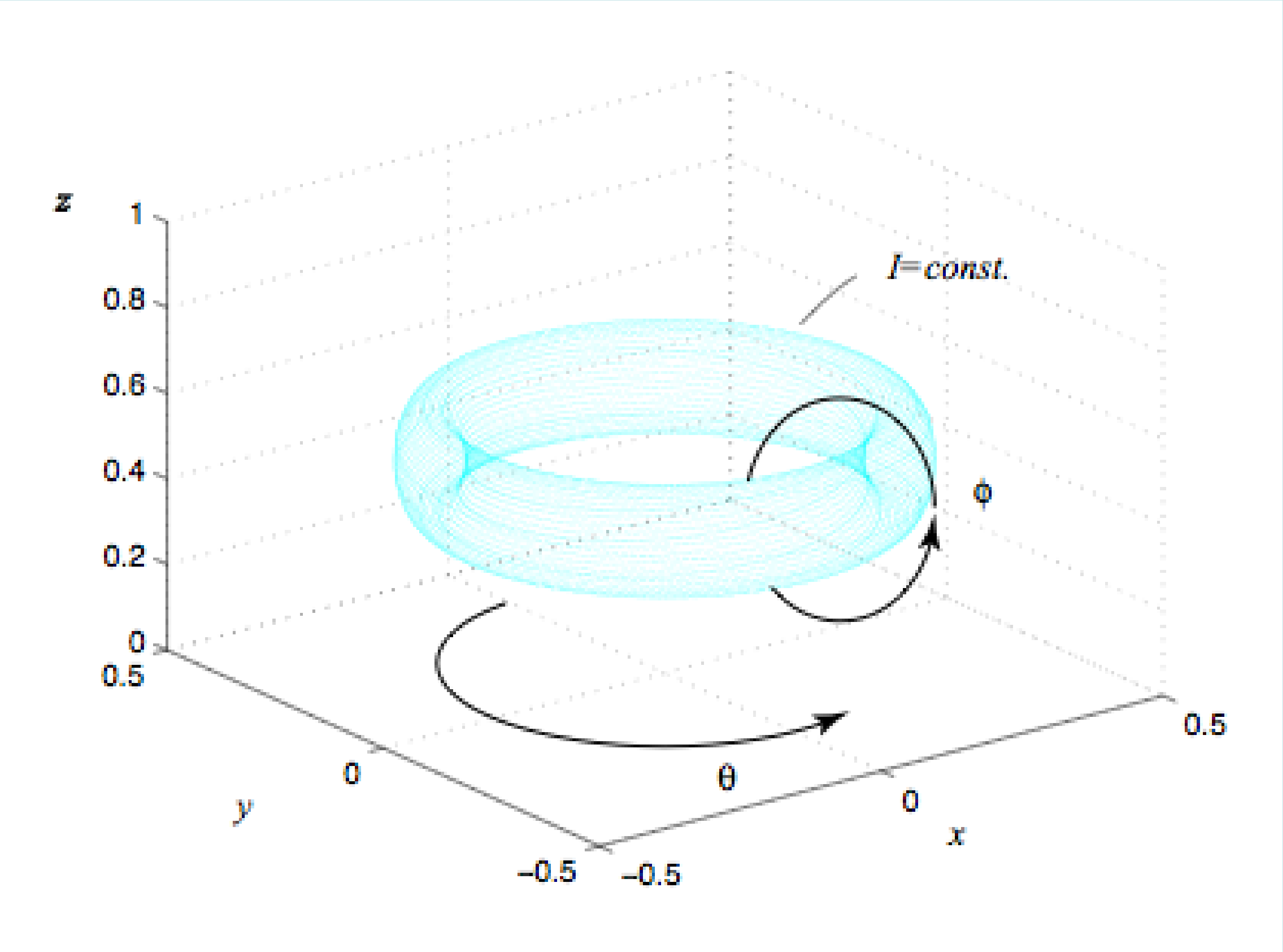


Action-Angle-Angle System

$$\frac{d\theta}{dt} = \Omega_{\theta}(I)$$

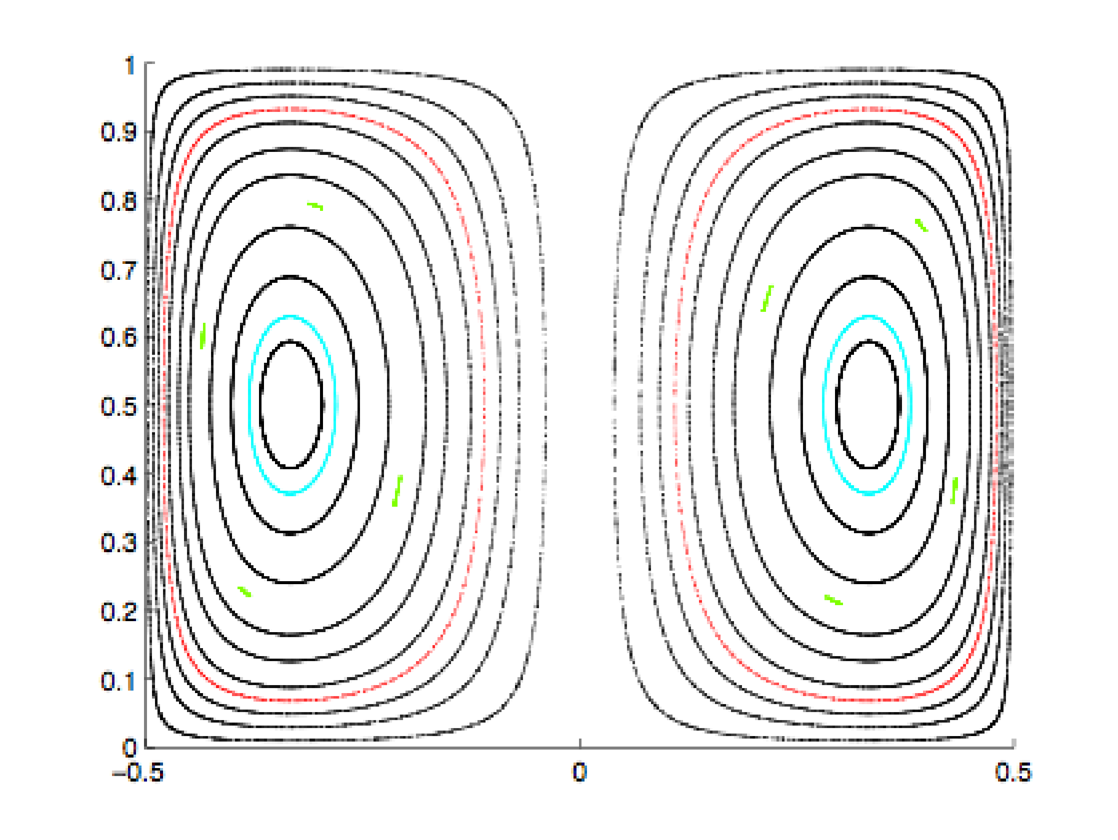
$$\frac{d\phi}{dt} = \Omega_{\phi}(I)$$

$$\frac{dI}{dt} = 0$$



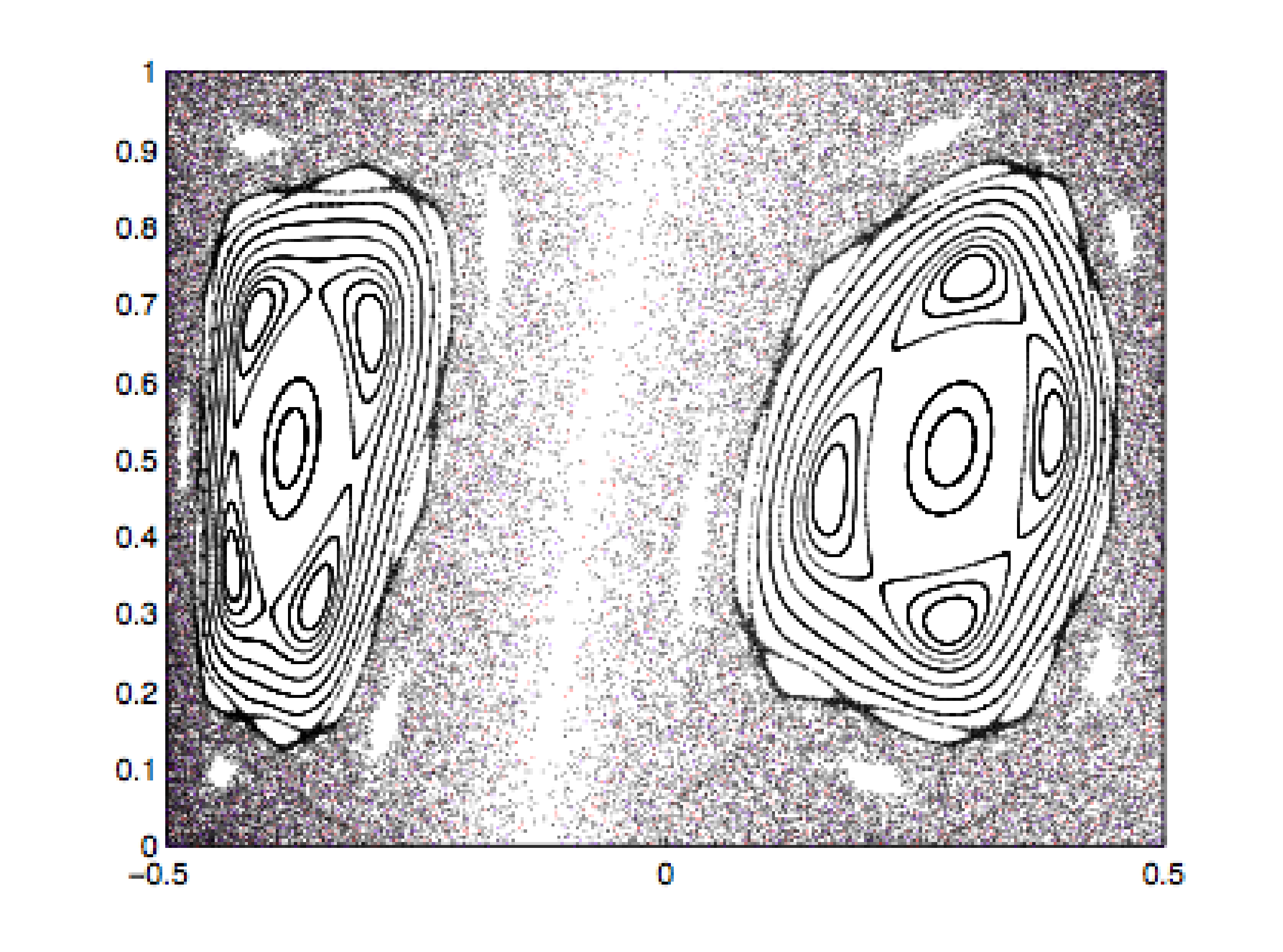
Mezic and Wiggins, 1994; Fountain, 2000

Steady case with no θ-depedence.



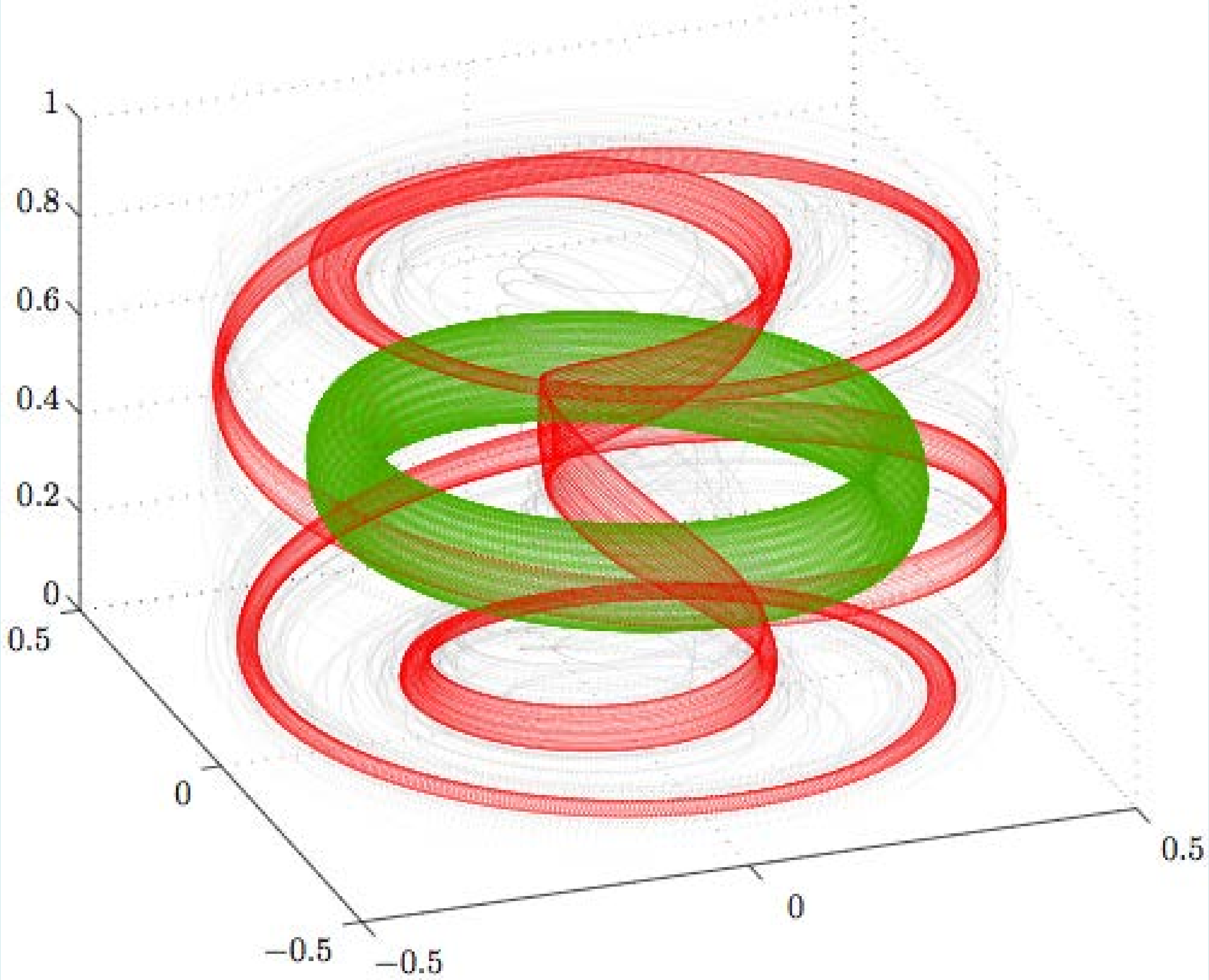
Add a steady, non-axisymmetric perturbation.

$$\frac{\Omega_\phi}{\Omega_\theta} = \frac{m}{n} \quad \text{(number of islands=}n\text{)}$$

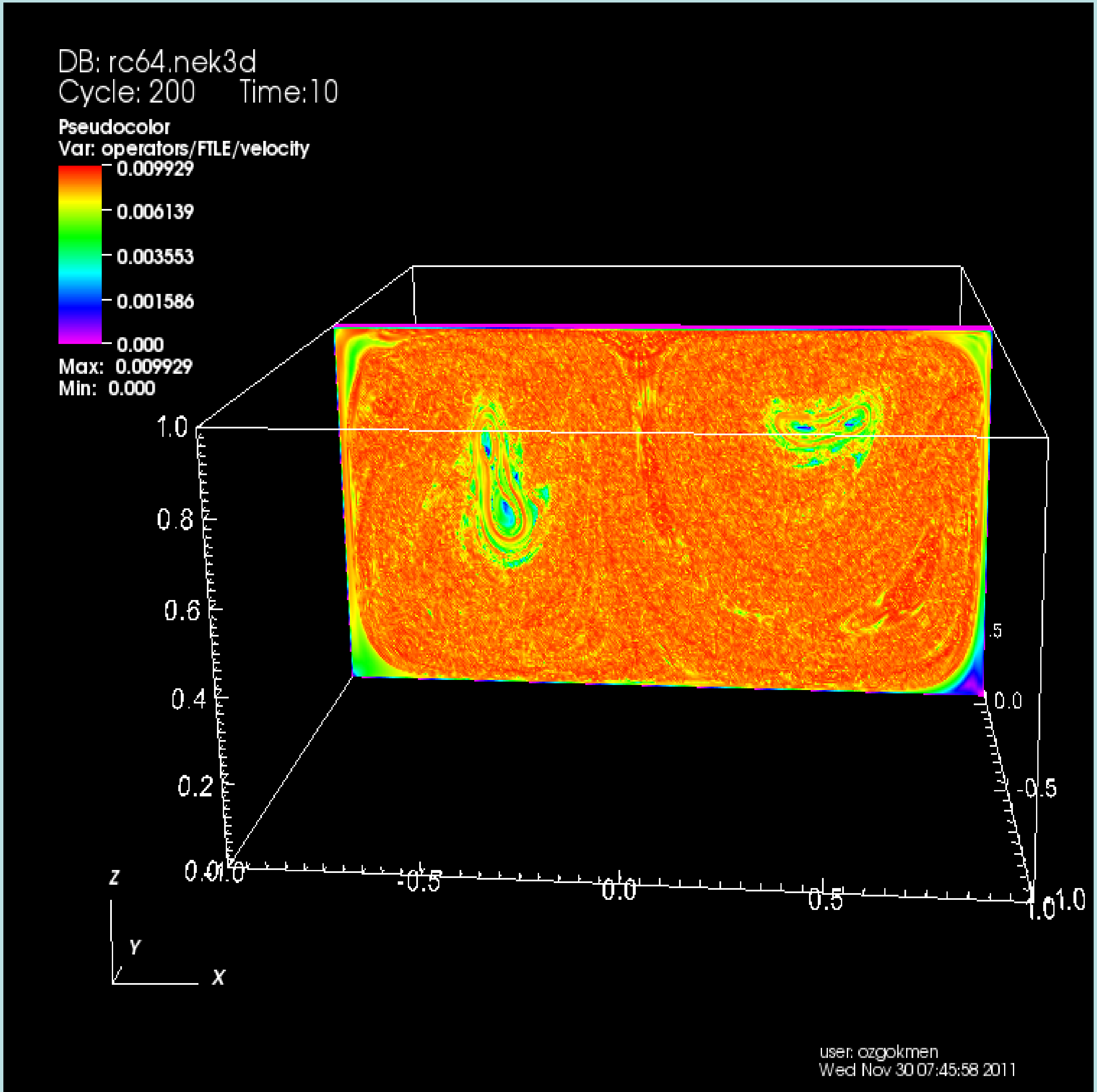


Poincare' Section

Examples of surviving tori.

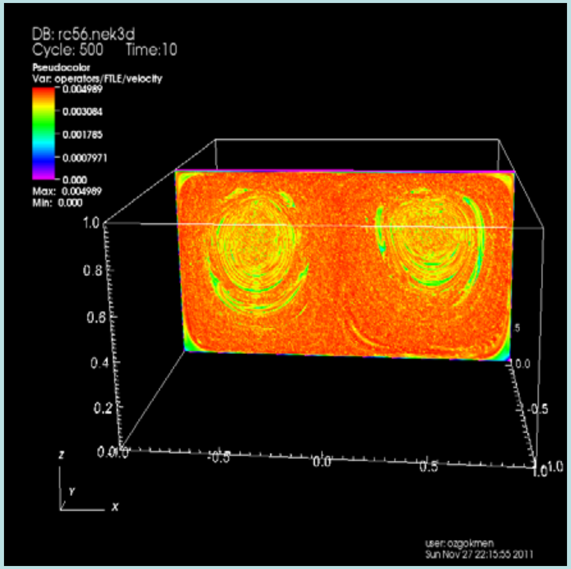


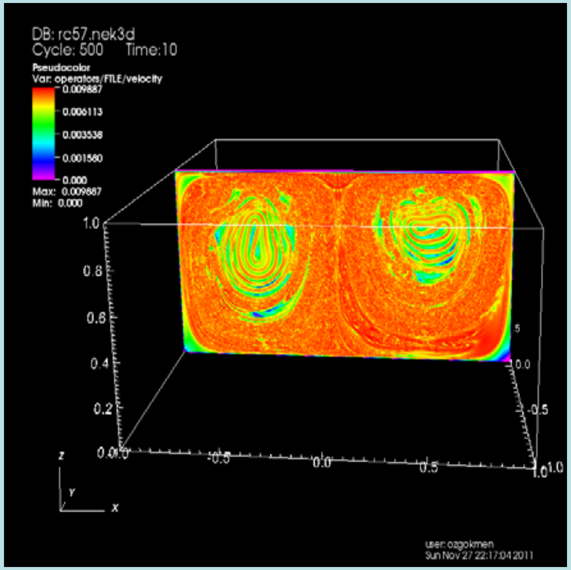
Numerical Simulation



Sensitivity to perturbation amplitude x_0 for E=1/20

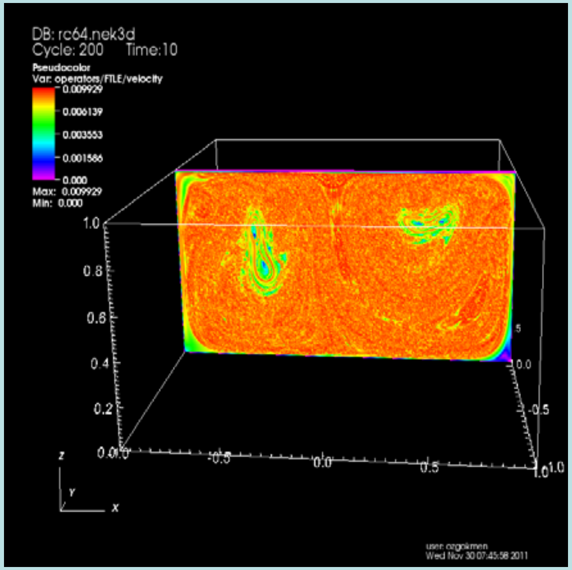
rc56, x_0=-0.01

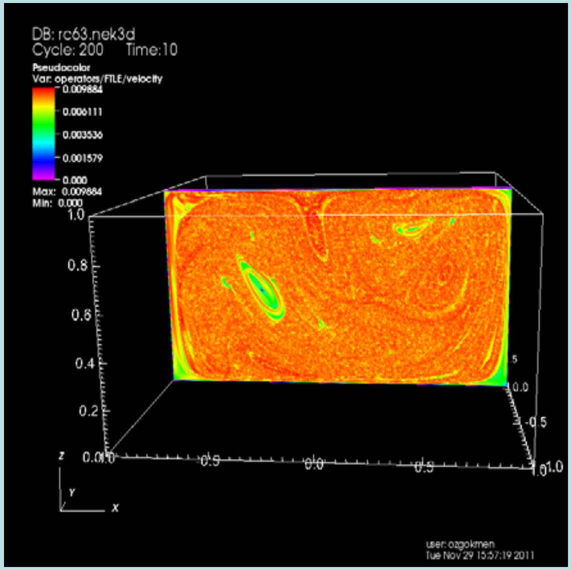




rc57, x_0=-0.02

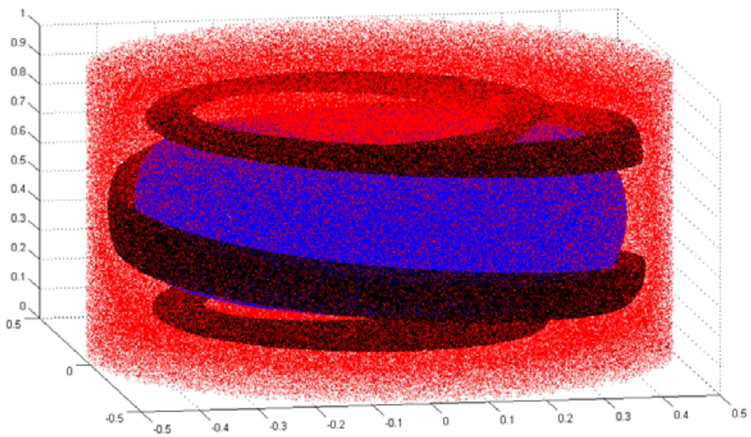
rc64, x_0=-0.05



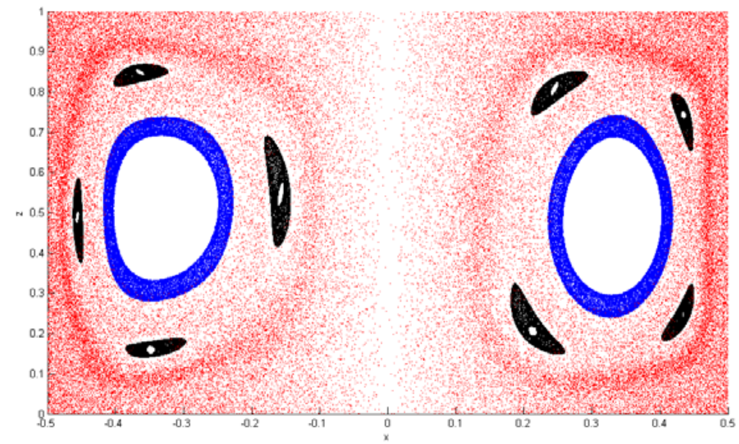


rc63, x_0=-0.15

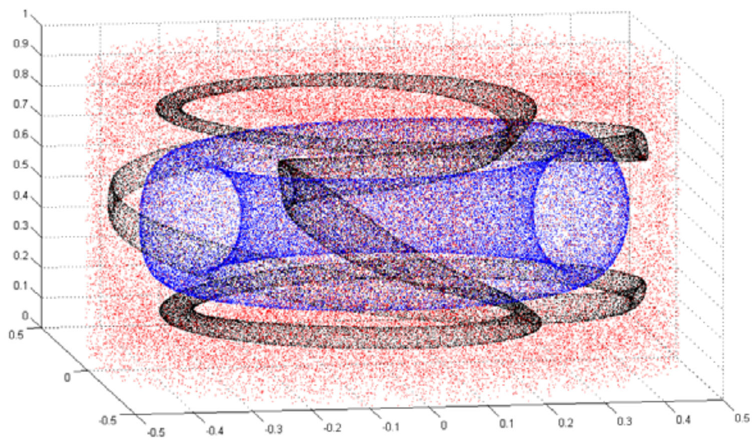
Perturbation with θ-dependence and time-dependence.



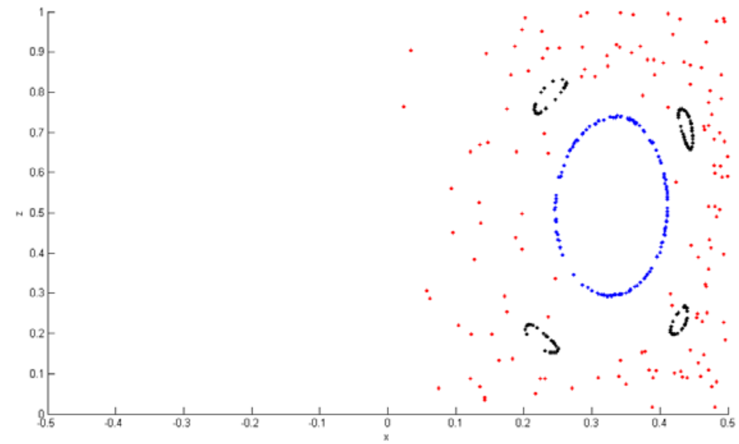
strobed trajectories



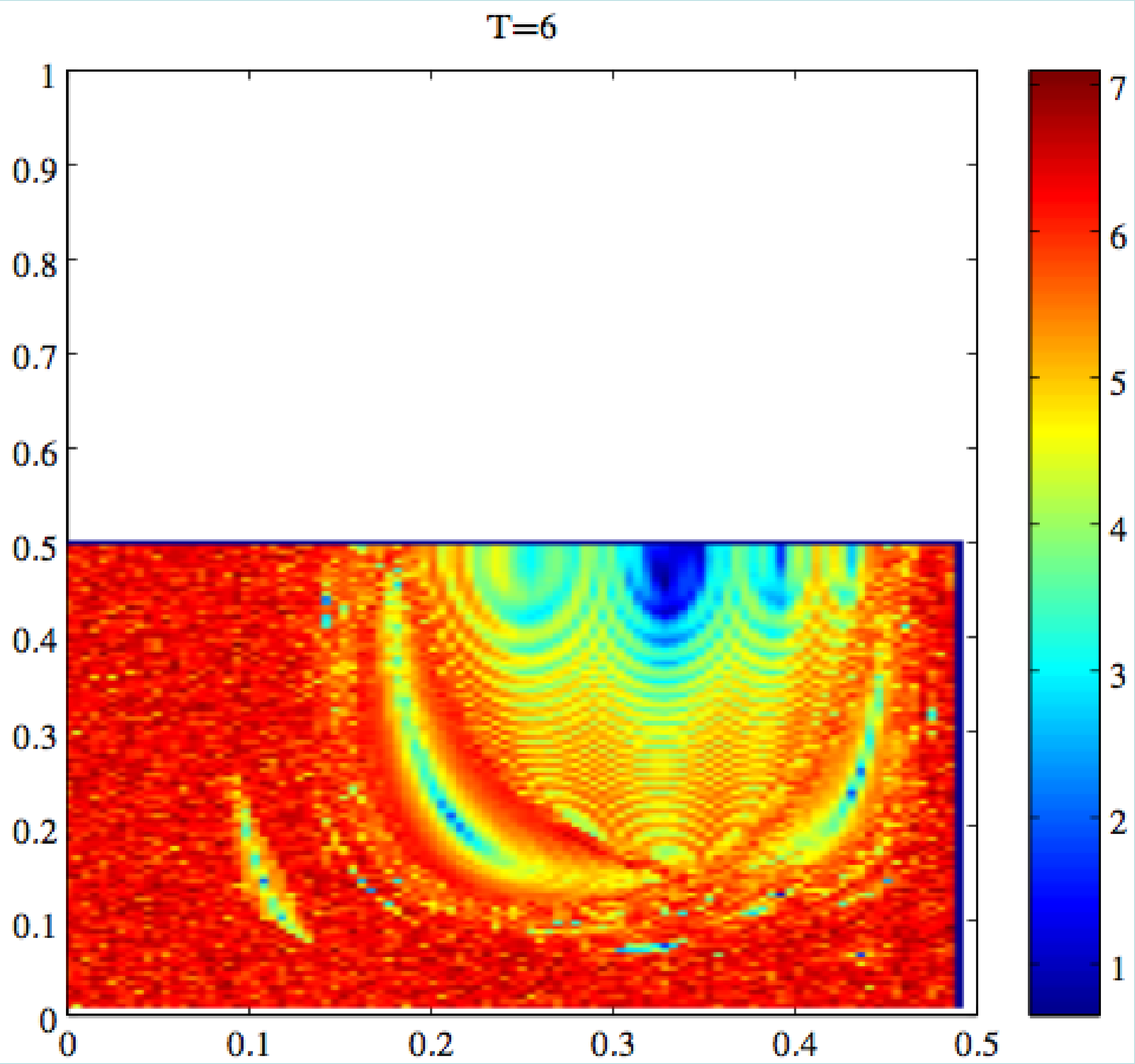
Poincare' map in θ only

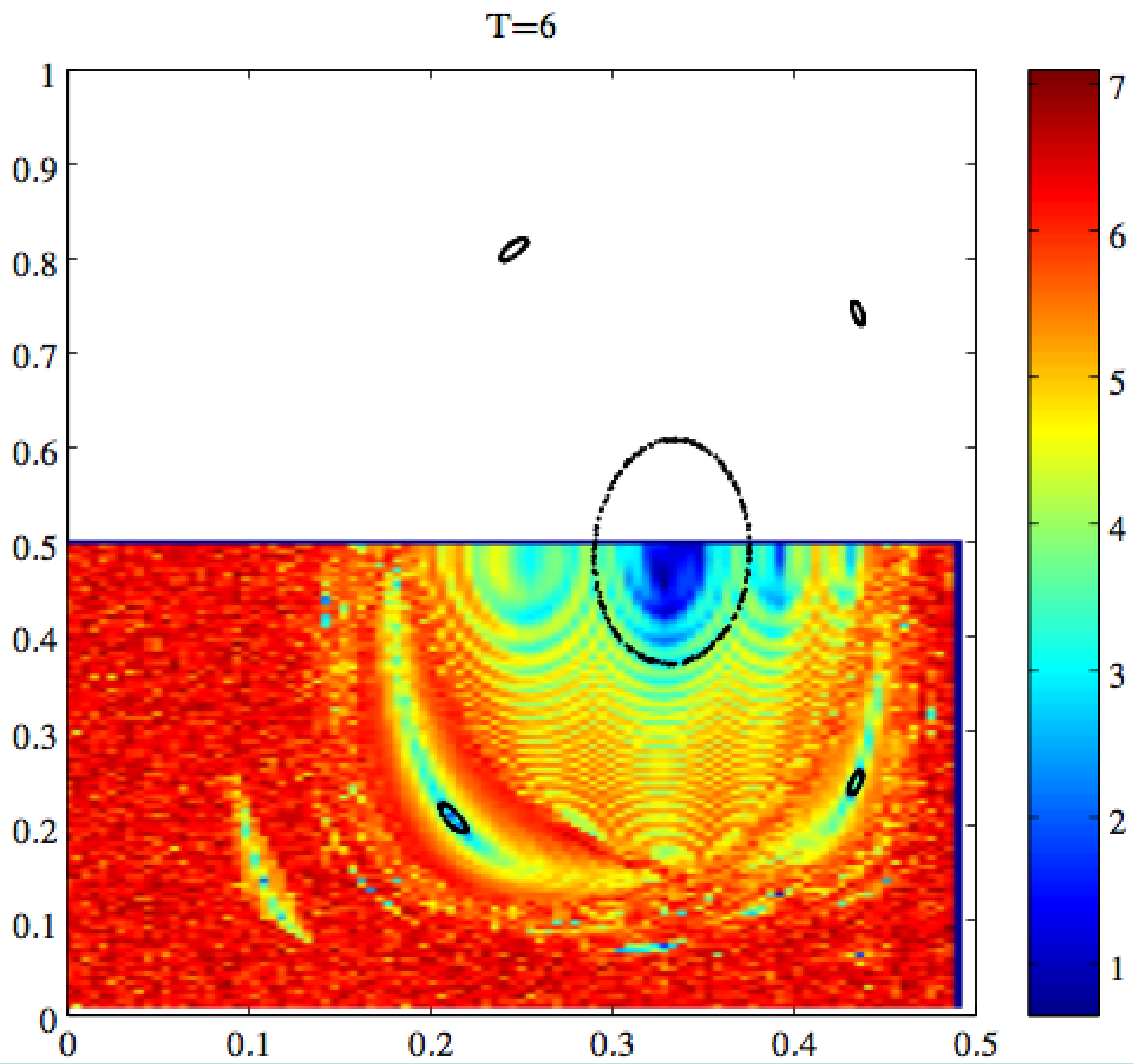


Snap shot of time-dependent tori

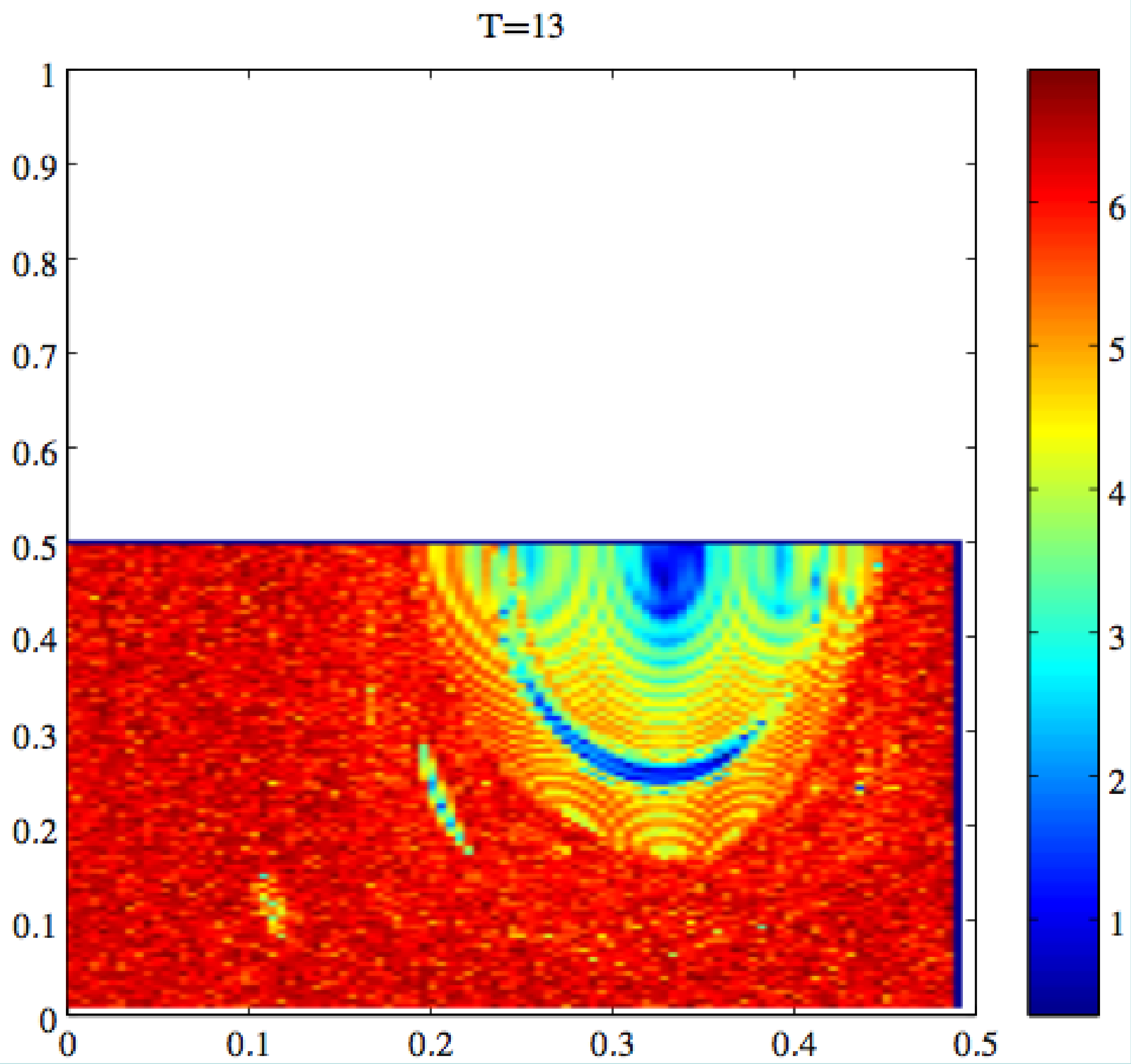


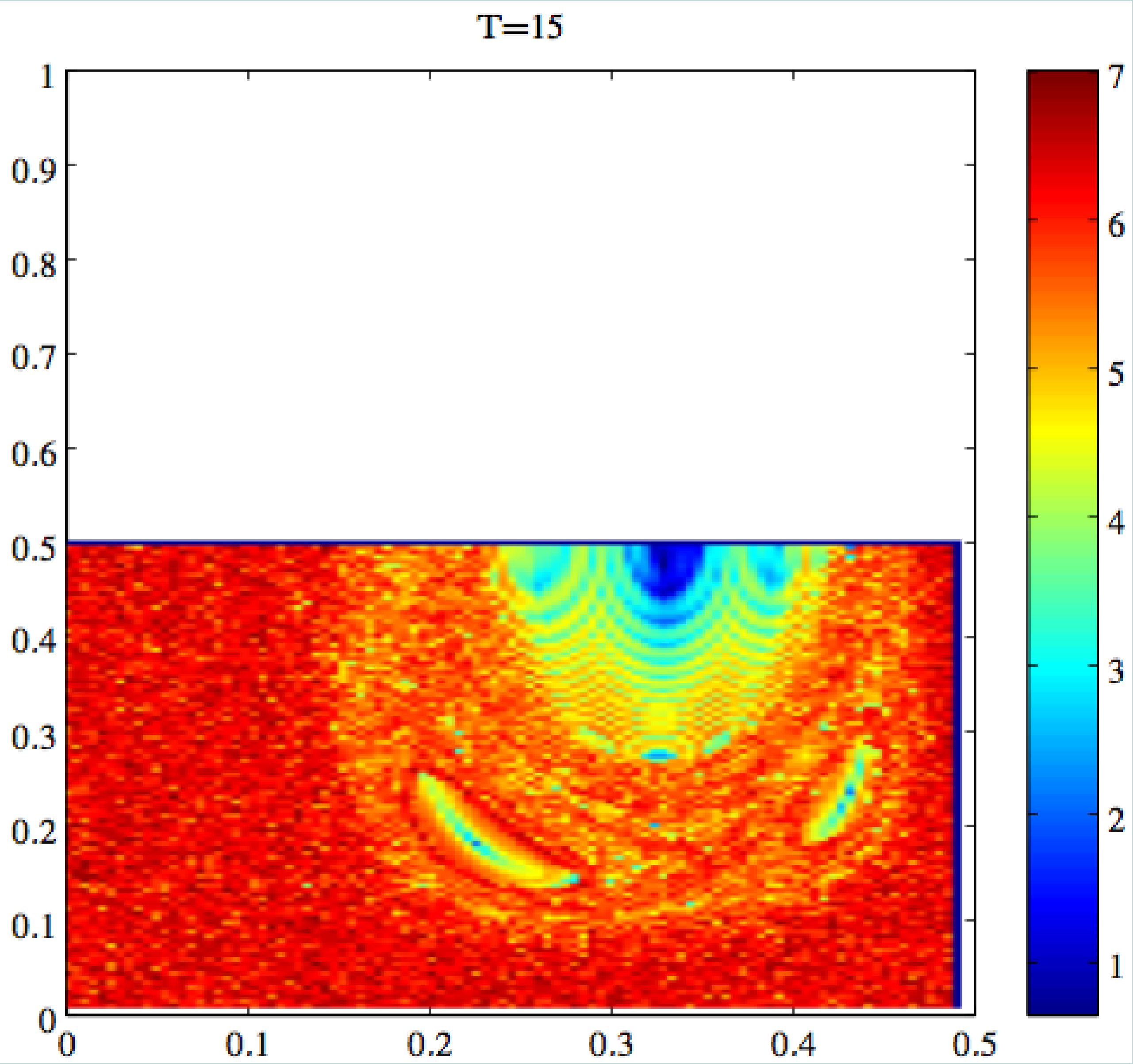
Double Poincare' map


T=6
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
7
6
5
4
3
2
1



T=6
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
7
6
5
4
3
2
1


T=13
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
6
5
4
3
2
1


T=15
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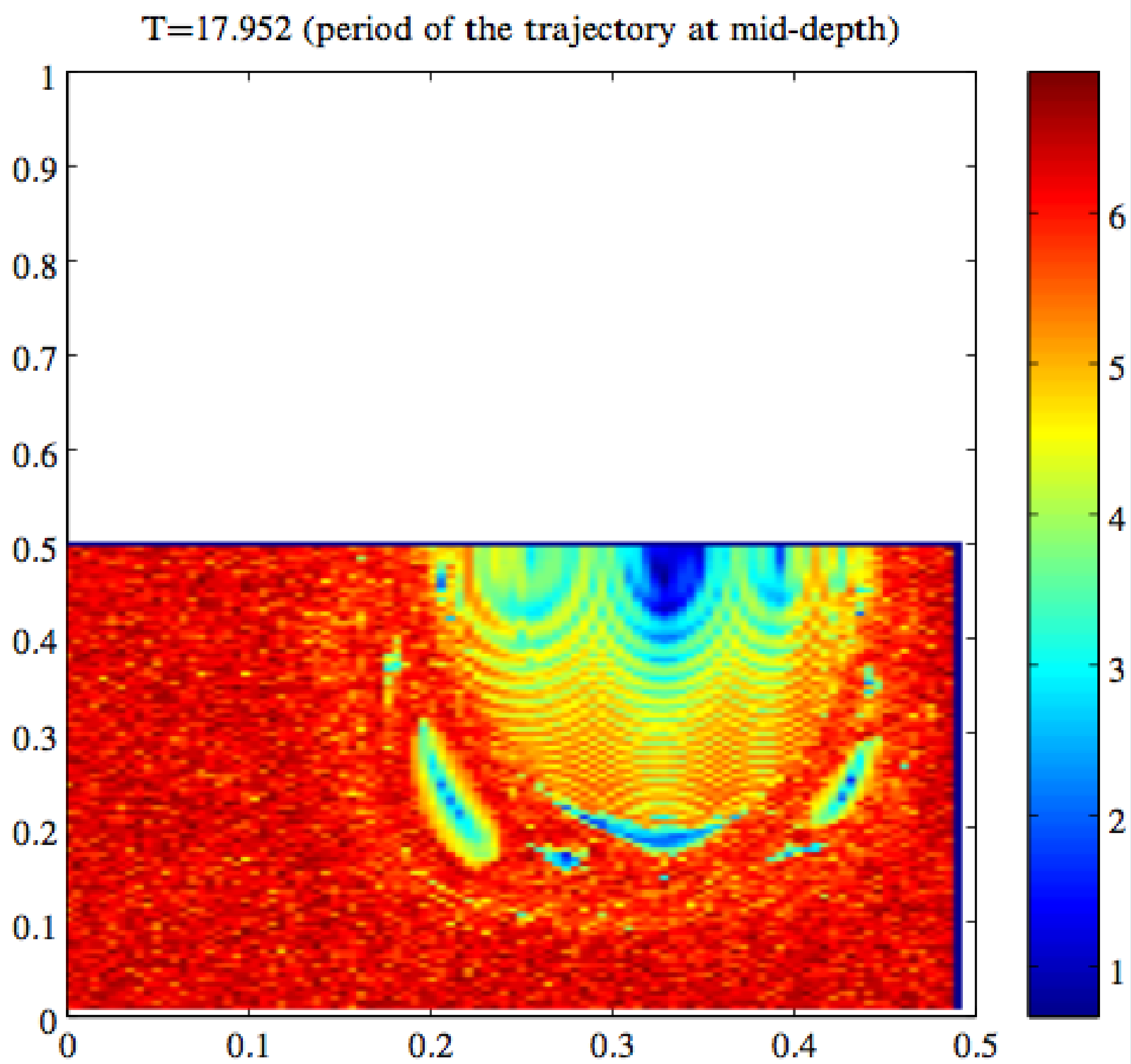
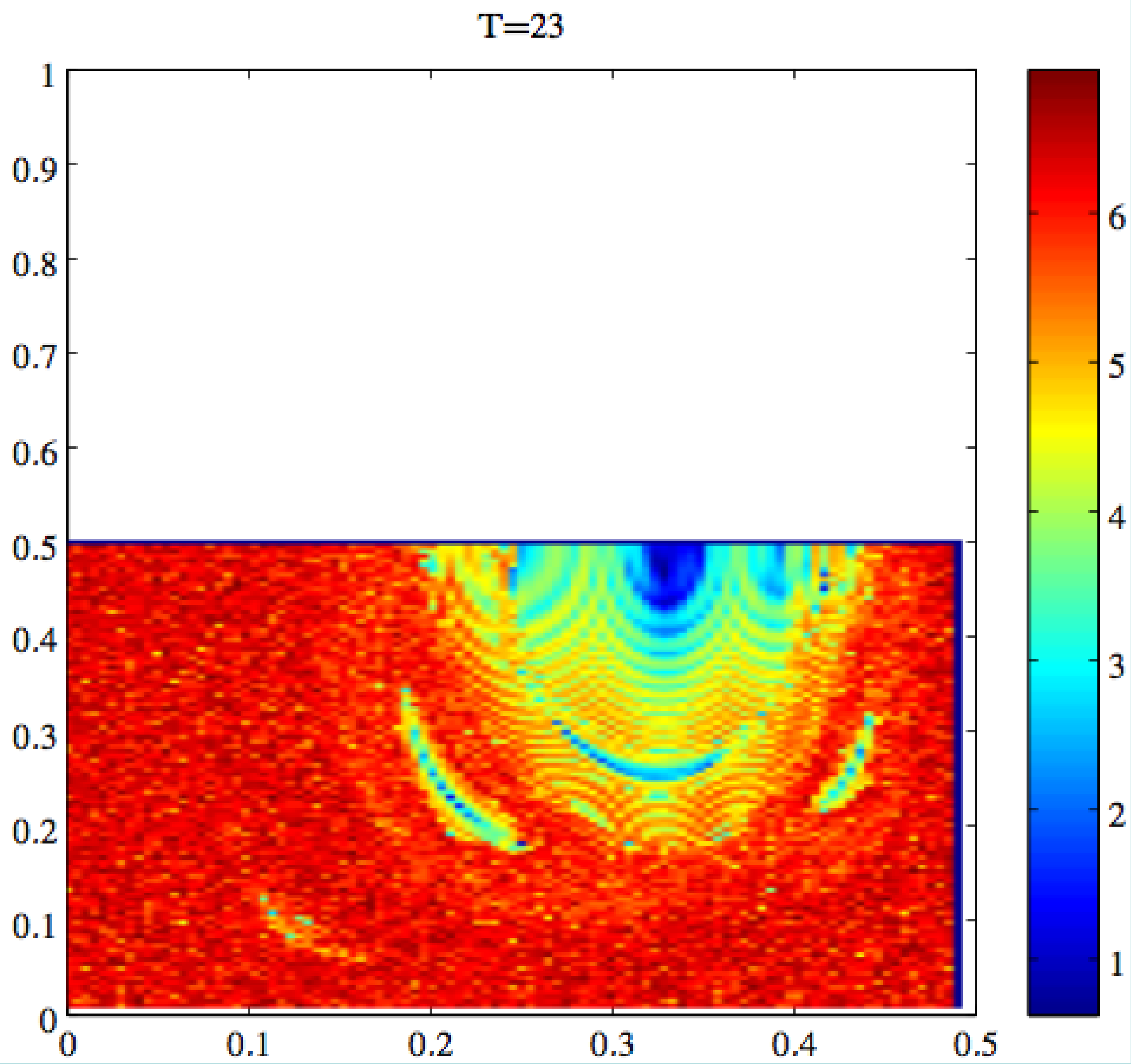
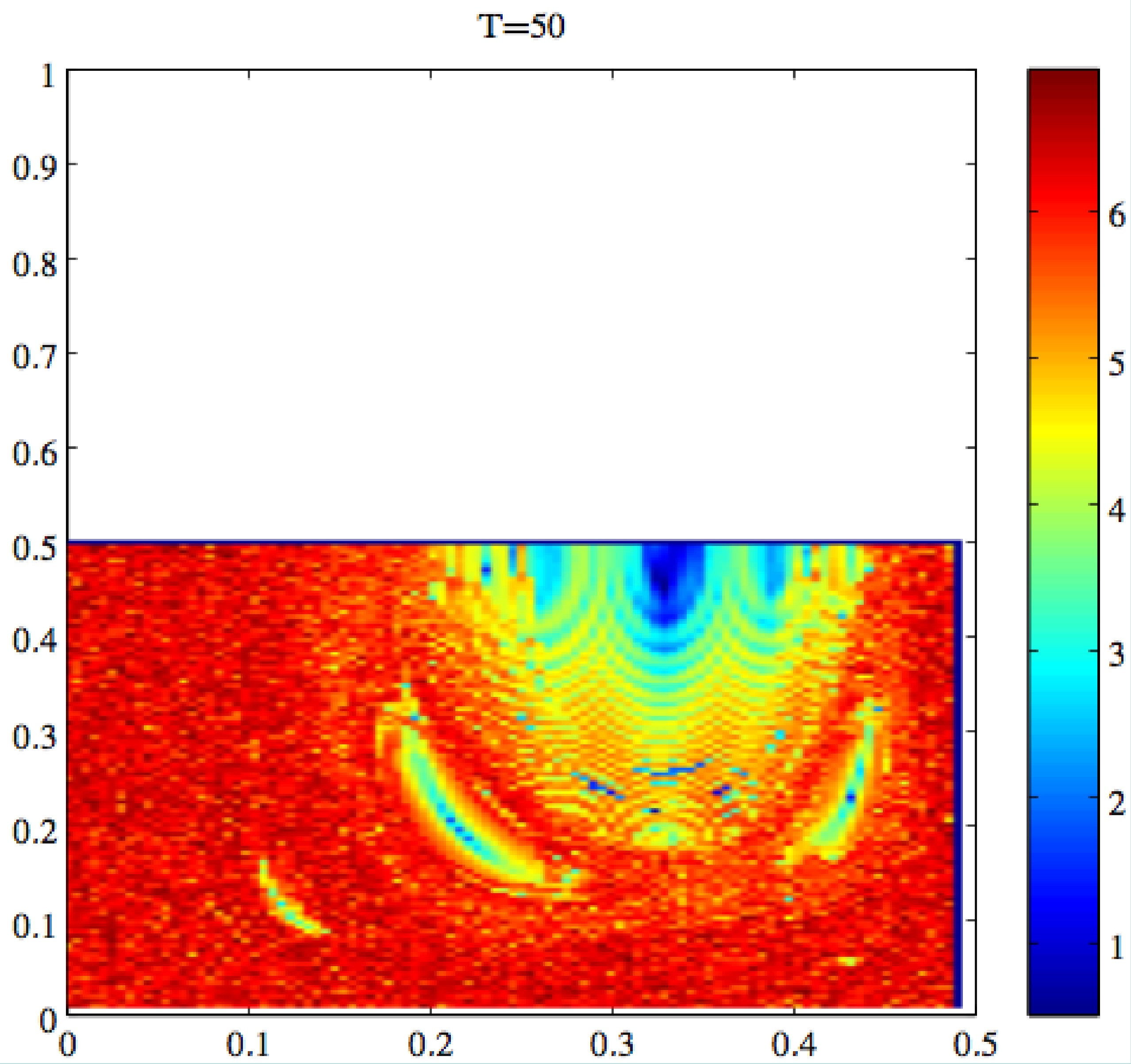
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
7
6
5
4
3
2
1



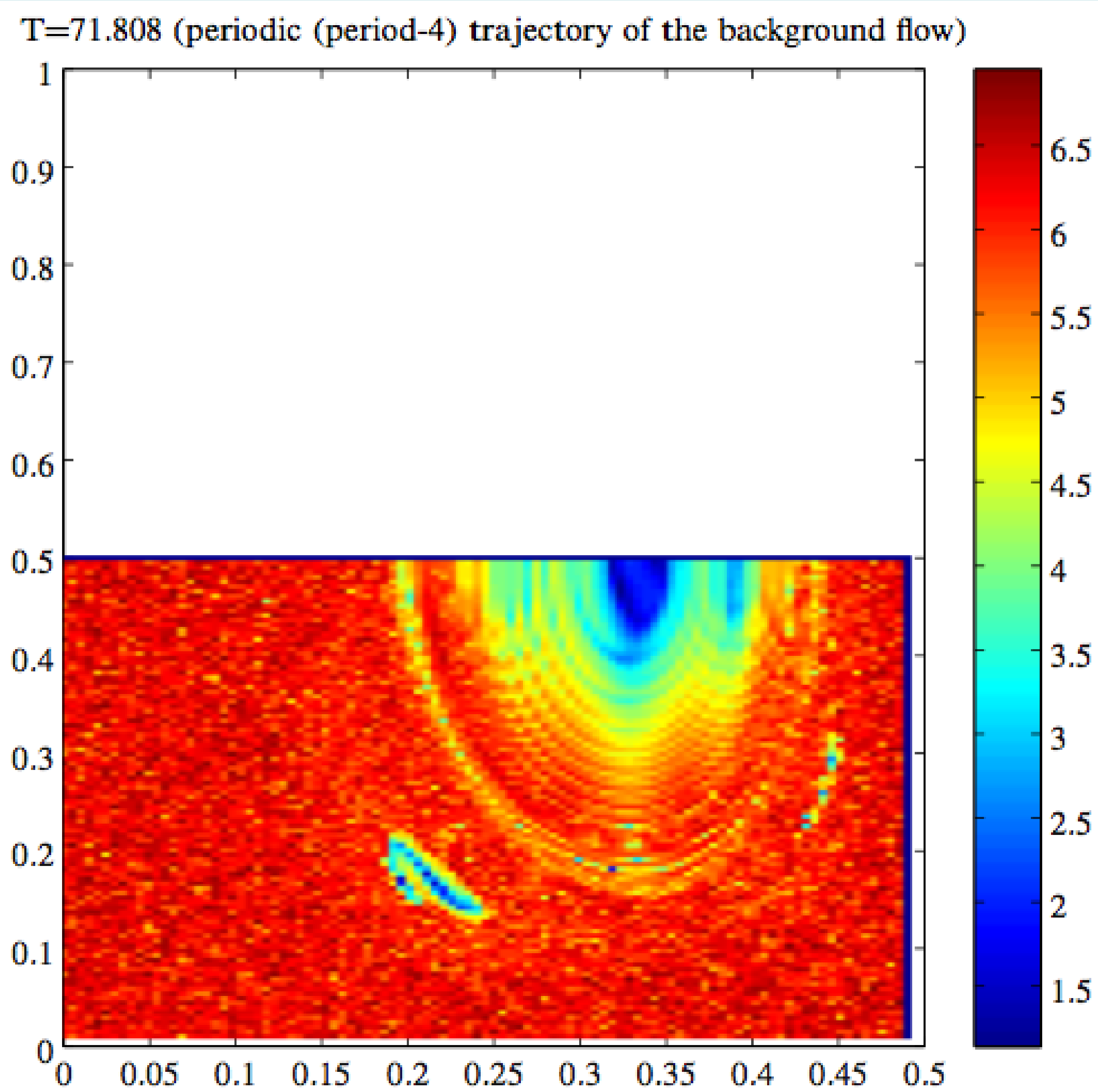
T=17.952 (period of the trajectory at mid-depth)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.1
0.2
0.3
0.4
0.5
1
2
3
4
5
6

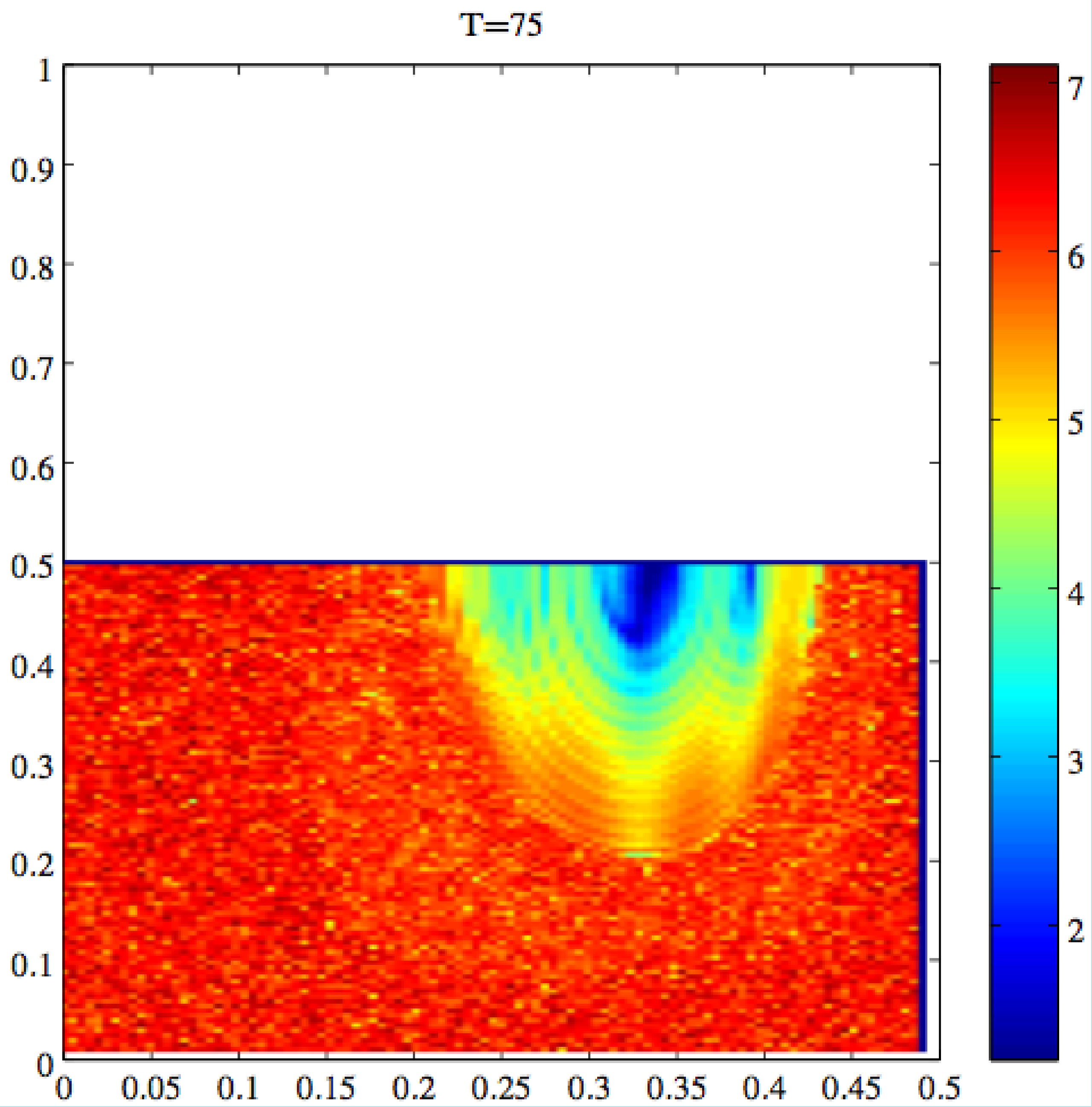


T=23
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
6
5
4
3
2
1



T=50
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
6
5
4
3
2
1


T=71.808 (periodic (period-4) trajectory of the background flow)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5


T=75
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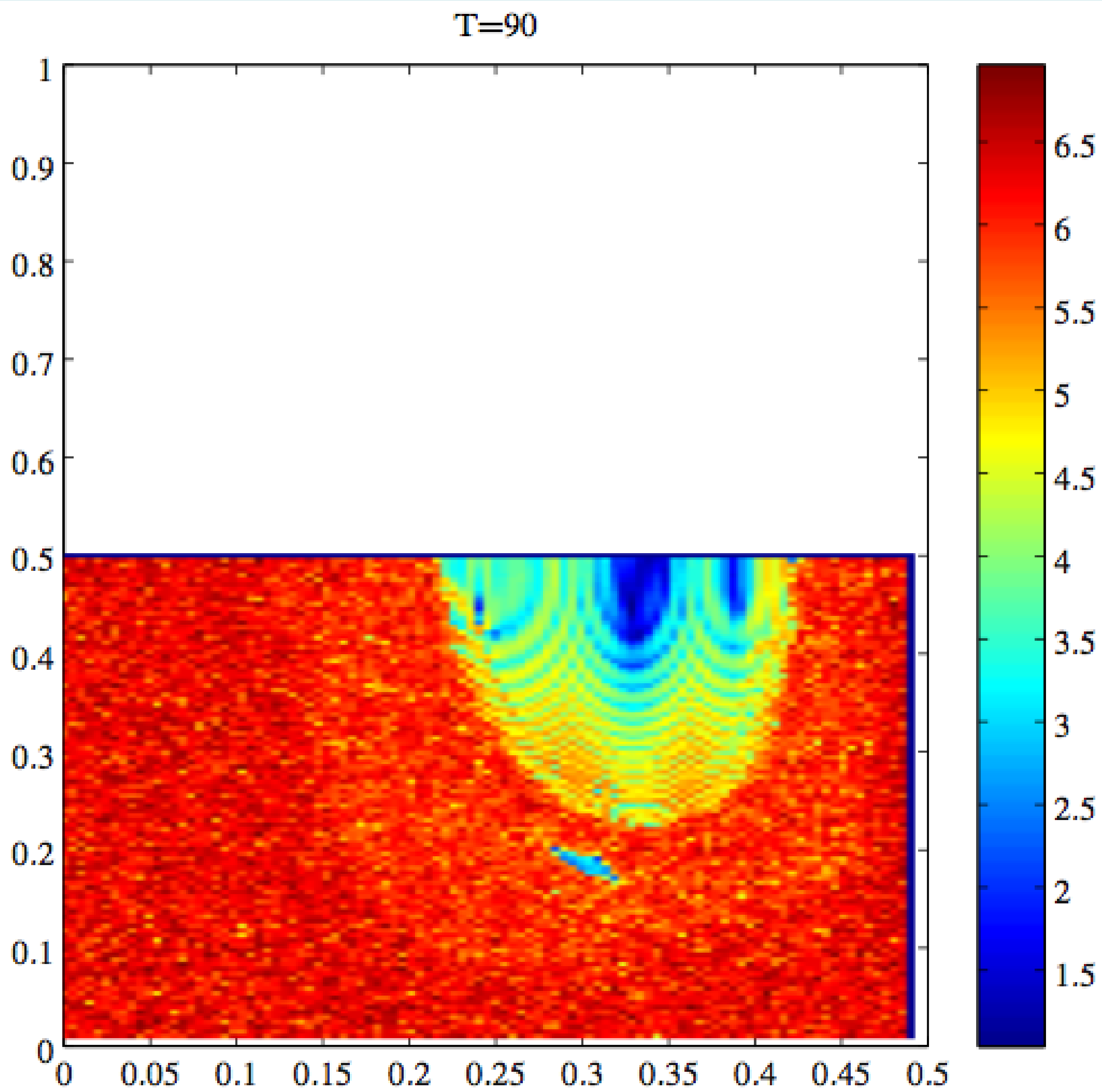
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
7
6
5
4
3
2


T=90
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5

Where is the Real Ocean?

Mesoscale eddies? Upwelling rates ($7<w<40$ cm/day) so overturning times are measured in years.

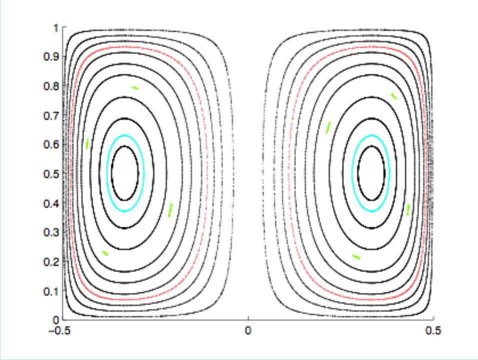
Sub-mesoscale? W is larger but do eddies live long enough For our mechanism to be relevant?

Langmuir circulations?

Hurricanes?

Where are the dynamics in ‘dynamical’ systems?

dynamics and thermodynamics



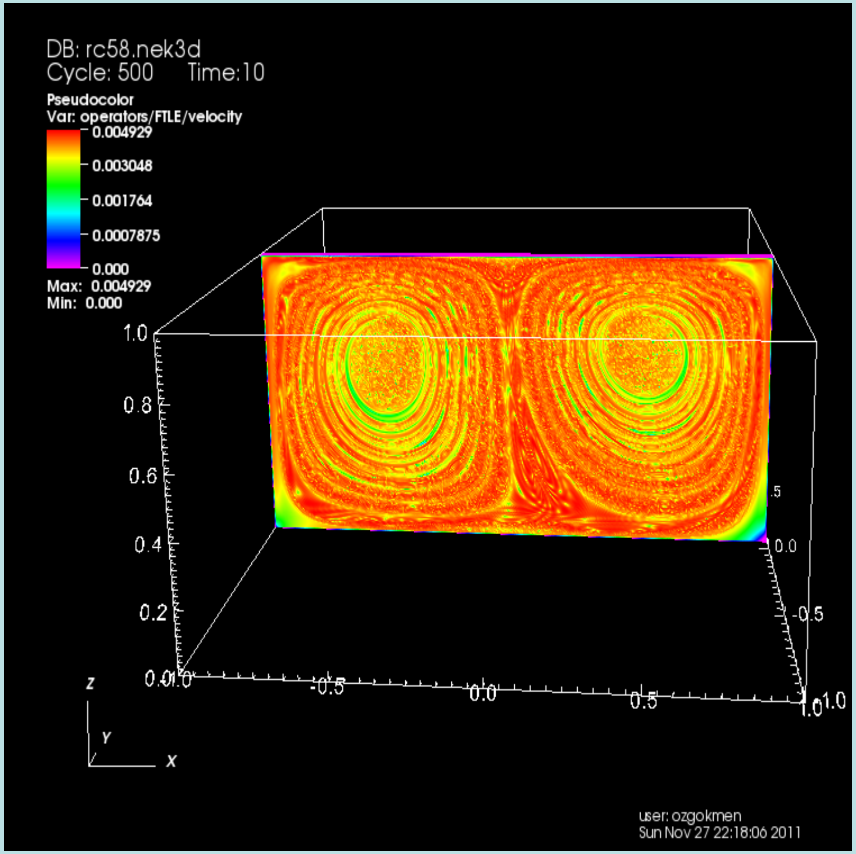
geometry

KAM theorem (resonance)

template for stirring/mixing

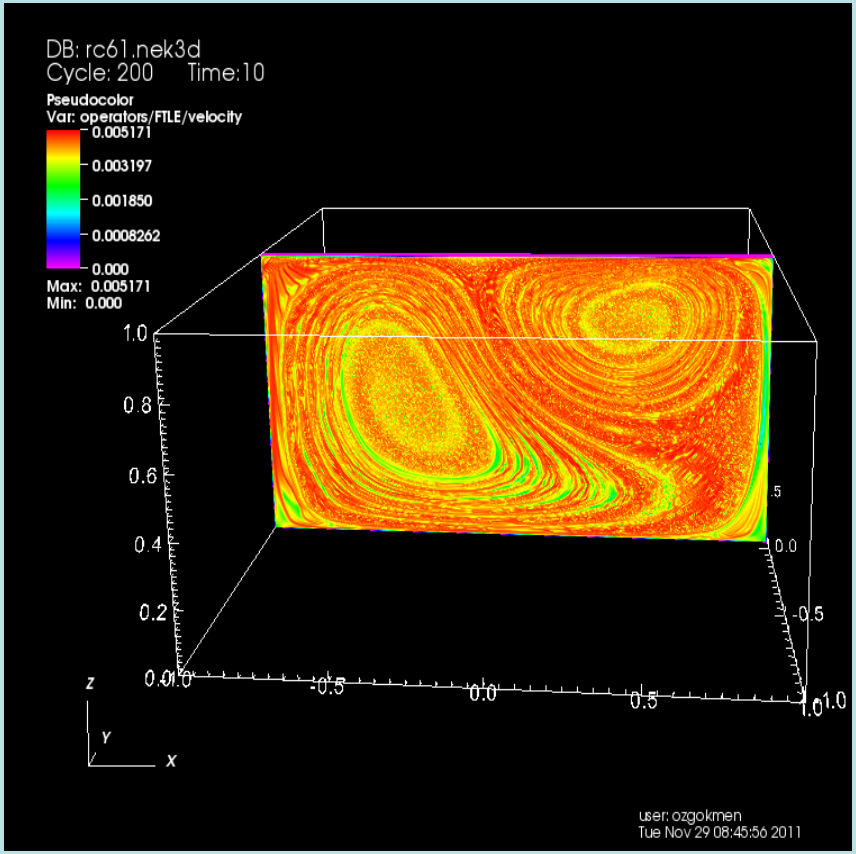
Finite-Time Lyapunov Exponents: *E*=1/6.5, large Rossby #

strong tilt



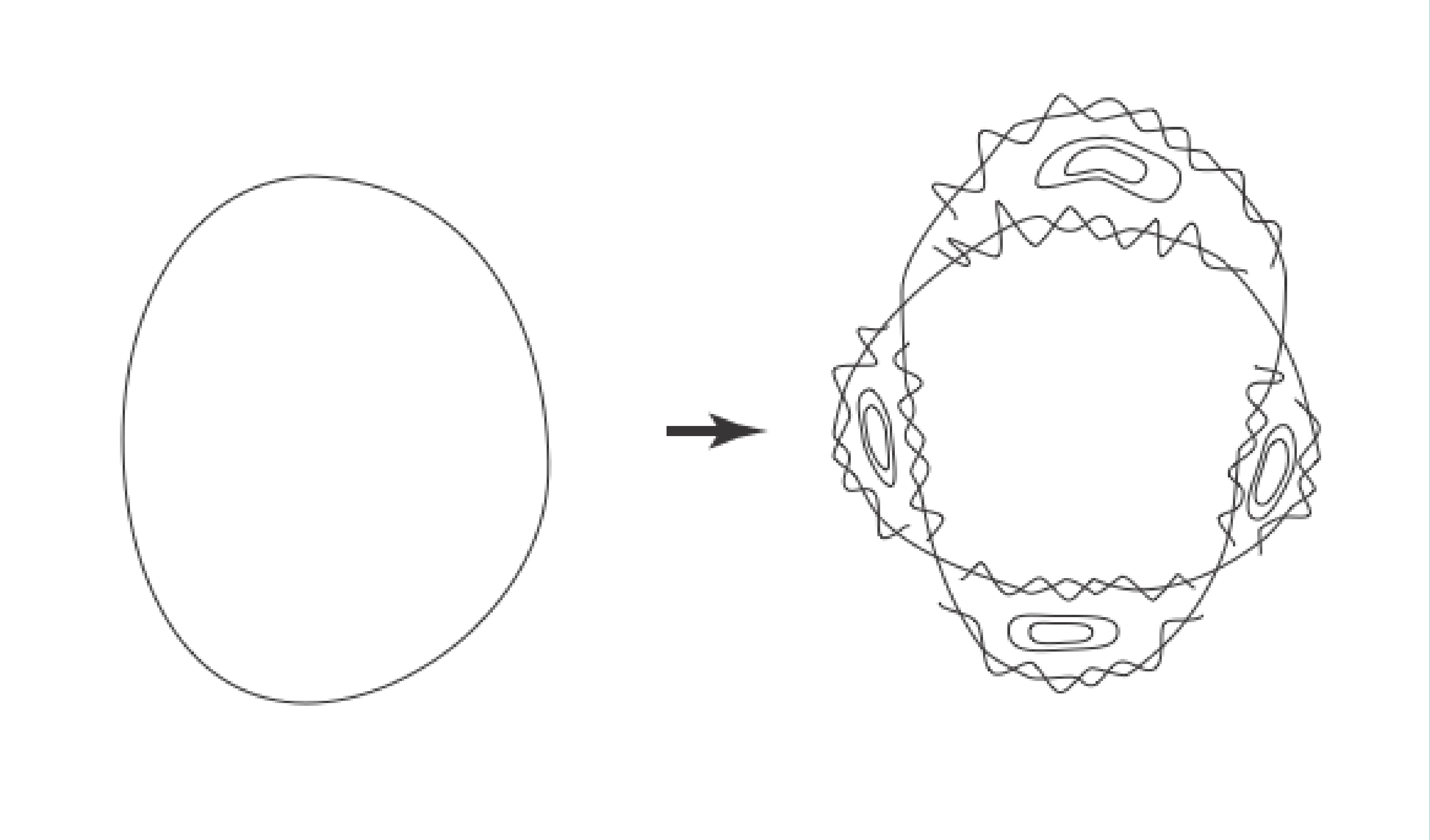
rc58, x_0=-0.02, T=1000

weak tilt



rc61, x_0=-0.15, T=1000

Breakup of torus with $\dfrac{\Omega_\phi}{\Omega_\theta} = \dfrac{m}{n}$ and $n = 4$.



Steady Perturbation+Time-periodic perturbation

$$\frac{dr}{dt} = \frac{1}{r}\frac{\partial \psi(r,z)}{\partial z} + \varepsilon u(x,y,z,t)$$

$$\frac{dz}{dt} = -\frac{1}{r}\frac{\partial \psi(r,z)}{\partial r} + \varepsilon w(x,y,z,t)$$

$$\frac{d\theta}{dt} = \frac{1}{r}\left(V(r) + \varepsilon v(x,y,z,t)\right)$$

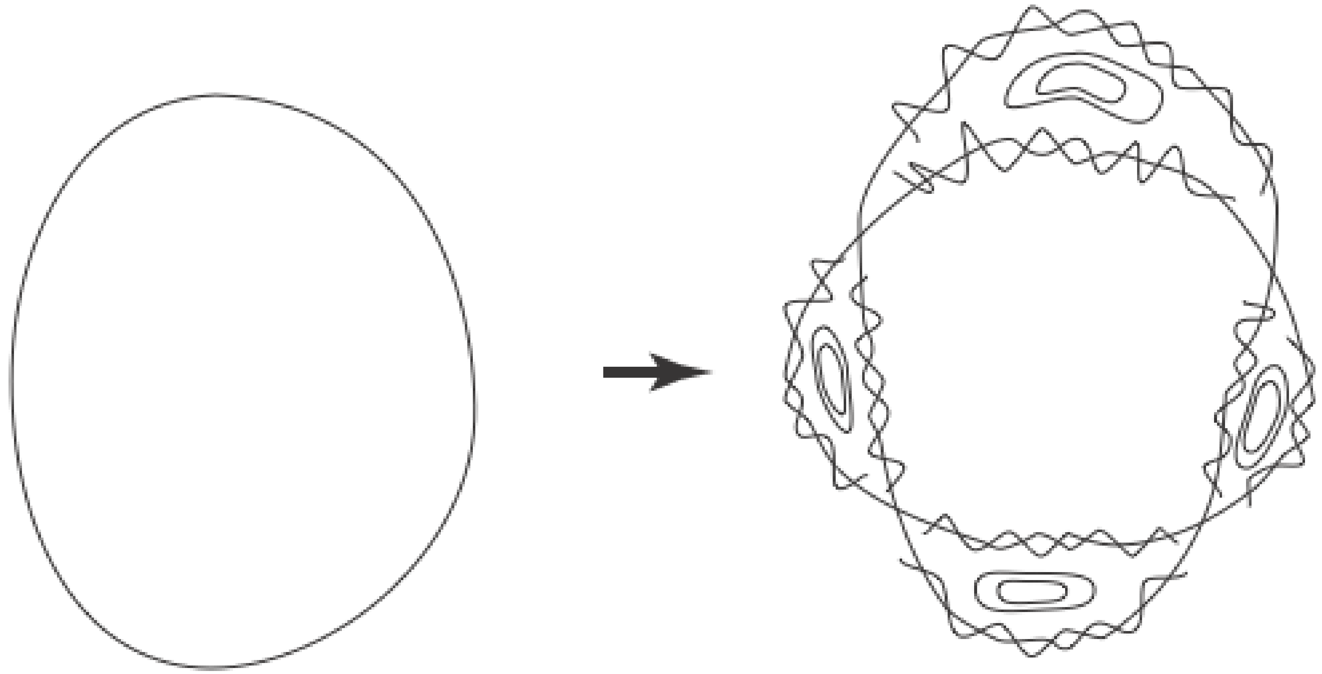
$$\frac{d\theta}{dt} = \Omega_\theta(I) + \varepsilon f_\theta(I,\theta,\phi,t)$$

$$\frac{d\phi}{dt} = \Omega_\phi(I) + \varepsilon f_\phi(I,\theta,\phi,t)$$

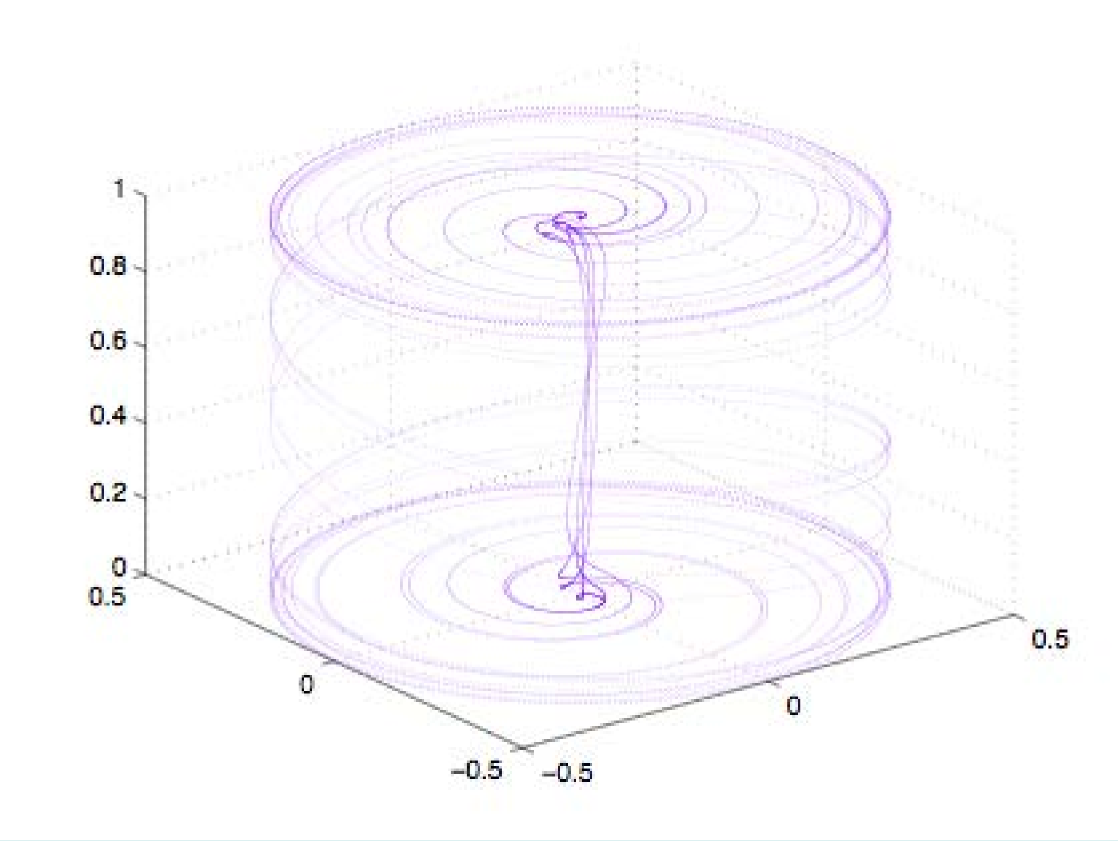
$$\frac{dI}{dt} = \varepsilon f_I(I,\theta,\phi,t)$$

Solomon and Mezic 2003: All barriers can be destroyed if forcing period is “close to typical circulation time”.

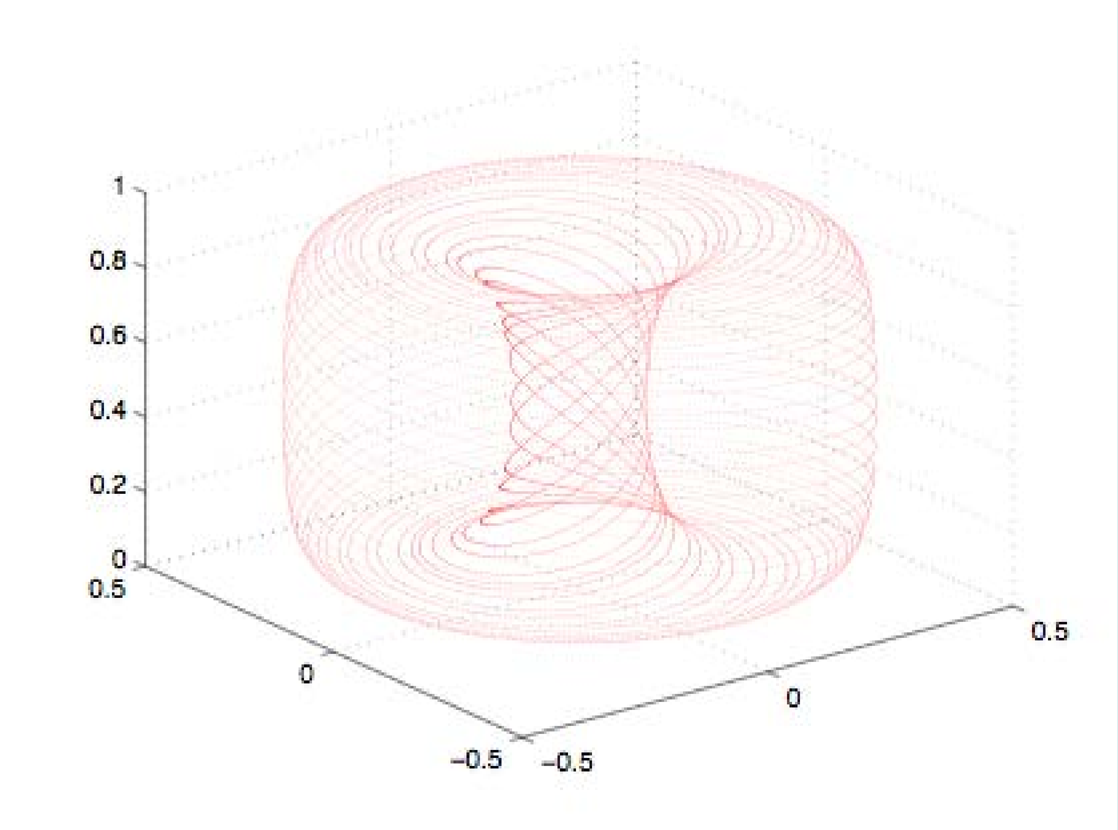
Breakup of torus with $\dfrac{\Omega_\phi}{\Omega_\theta} = \dfrac{m}{n}$ and $n = 4$.



Steady flow with no dependence on azimuth θ.



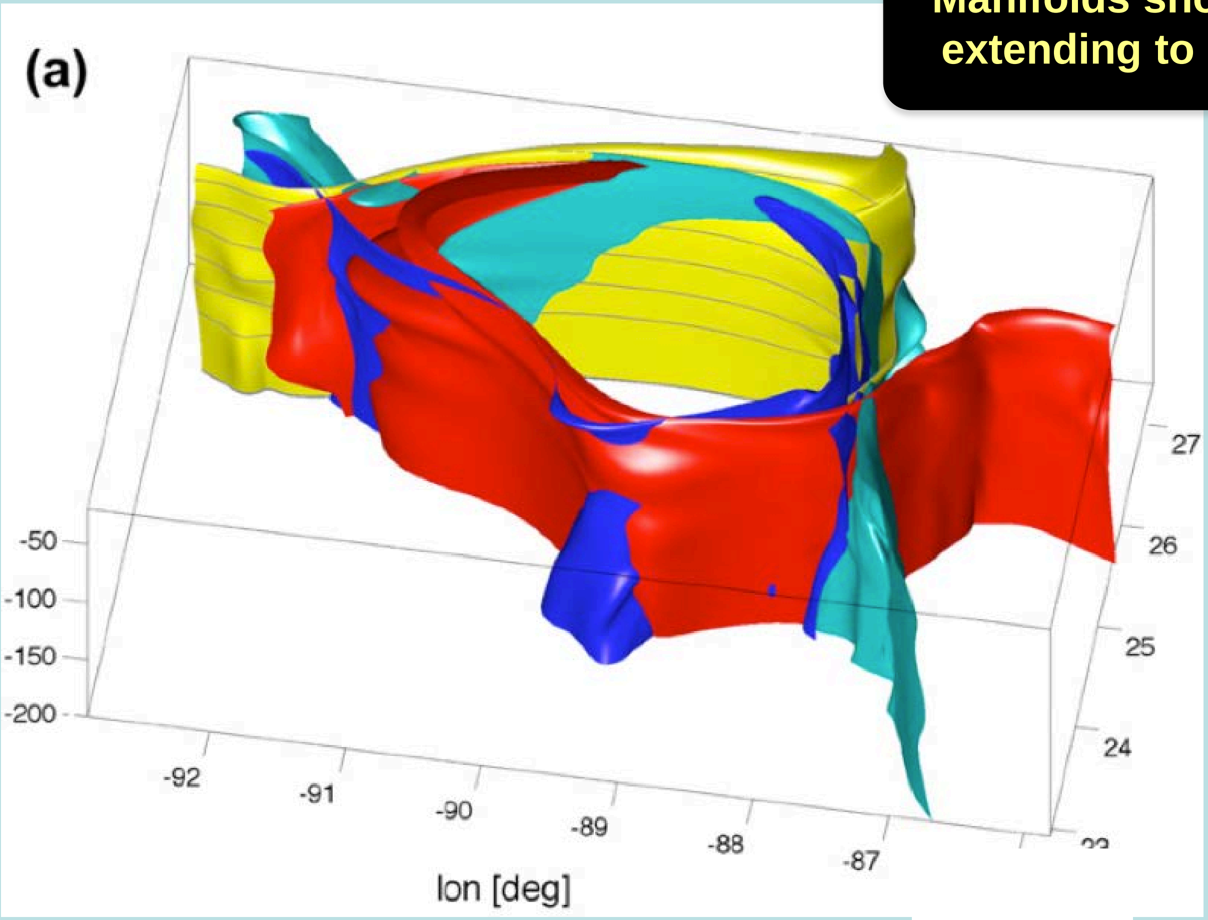
Each trajectory lives on a torus.



A 3D View of Loop Current Ring Exchange

(Branicki and Kirwan, 2010)

Manifolds show mixing boundaries extending to depths of 200 meters



Steady Perturbation

$$\frac{dr}{dt} = \frac{1}{r}\frac{\partial \psi(r,z)}{\partial z} + \varepsilon u(x,y,z)$$

$$\frac{dz}{dt} = -\frac{1}{r}\frac{\partial \psi(r,z)}{\partial r} + \varepsilon w(x,y,z)$$

$$\frac{d\theta}{dt} = \frac{1}{r}\big(V(r) + \varepsilon v(x,y,z)\big)$$

$$\frac{d\theta}{dt} = \Omega_\theta(I) + \varepsilon f_\theta(I,\theta,\phi)$$

$$\frac{d\phi}{dt} = \Omega_\phi(I) + \varepsilon f_\phi(I,\theta,\phi)$$

$$\frac{dI}{dt} = \varepsilon f_I(I,\theta,\phi)$$

If $\dfrac{\Omega_\phi}{\Omega_\theta} = \dfrac{m}{n}$ (rational) the torus $I = const$ will break up.

If $\dfrac{\Omega_\phi}{\Omega_\theta}$ is irrational the torus $I = const$ may survive.