Dynamically consistent weakly 3D transport using SQG

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Dynamical Systems Theory and Lagrangian Data Assimilation in 3D+1 Geophysical Fluid Dynamics

Why QG?

- The ocean is a thin fluid envelope on a rapidly rotating Earth. Dynamics are largely two-dimensional with weak vertical variation.
- Behavior is governed by the dimensionless Rossby number $Ro = U/Lf$, ratio of local vorticity to planetary vorticity. Small for large-scale motions.
- Leading-order asymptotic prognostic equations are the Quasi-Geostrophic equations. Conserve QGPV. Vertical velocity is $O(Ro)$ and dynamically related to buoyancy.

Why SQG?

- Ocean 3D+1 is investigating transport in 3D+1. Obtaining dynamically consistent 3D+1 velocity fields is generally expensive and time-consuming.
- SQG provides a system with dynamically consistent velocity fields that obeys a 2D equation for buoyancy on the boundary but have $O(Ro)$ vertical velocity in the interior.
- Can use existing simple dynamical models such as point vortices, moment truncations, etc... or move to turbulence in a periodic square/cube.

Quasigeostrophic (QG) Equations

Reduced equation of motion for $Ro \ll 1$

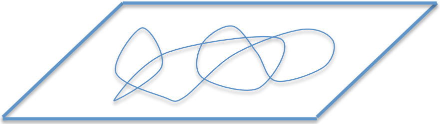
ζ	Vorticity
$\theta = f\partial_z\psi$	Buoyancy ($\sim$ density)
ψ	Streamfunction

bulk	$\partial_t\zeta = -J(\psi,\zeta) + f\partial_z w$
surface	$\partial_t\theta = -J(\psi,\theta) - N^2 w$
bulk	$q = \left[\partial_{xx} + \partial_{yy} + \partial_z\left(\frac{f}{N}\right)^2\partial_z\right]\psi$

Pedlosky (1982)

Surface Quasigeostrophic Equations

Surface QG (SQG) assumes potential vorticity $q = 0$ in the interior, so the dynamics are governed by the boundaries (usually just the surface).

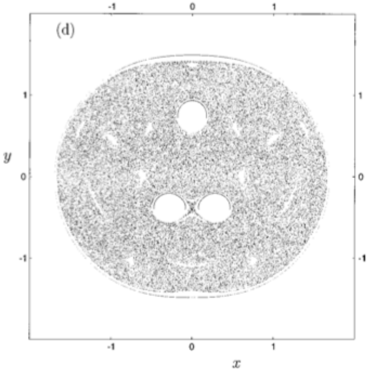


Some $\theta = f\,\frac{\partial \psi}{\partial z}$ distribution on the surface

Motion induced below by q=0

Understanding Vortex Behavior

Three classical point vortices (which have regular motion) induce chaotic motion in passive scalars. Look at transport properties of SQG point vortices.



Kuznetsov & Zaslavsky (1998)

Equations of Motion on the Surface

$$\theta_0 = \sum_n \kappa_n \delta(x - x_n)\delta(y - y_n)$$

$$(\dot{x}_n, \dot{y}_n) = \sum_{m \neq n} \frac{\kappa_m}{2\pi} \frac{1}{|\vec{x}_n - \vec{x}_m|^3}(-y_n + y_m, x_n - x_m)$$

Hamiltonian system:

$$\psi_0 = \frac{1}{2\pi|\vec{x} - \vec{x}_n|}$$

$$\vec{u}_0 = \sum_n \frac{\kappa_n}{2\pi|\vec{x} - \vec{x}_n|^3}(-y + y_n, x - x_n, 0)$$

Dynamically (asymptotically) consistent

Change of Variables

Aref & Pomphrey (1982) and Kuznetsov & Zaslavsky (1998)

$$z_j = \frac{1}{\sqrt{3}} \sum_{n=1}^{2} \sqrt{2J_n} e^{i\theta_n} e^{-2i\pi n(j-1)/3}$$

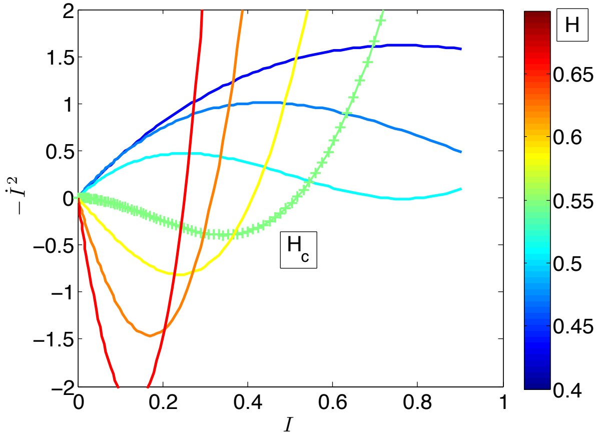
$$I = \left(\frac{J_2 - J_1}{J_2 + J_1} \right)^2 = 16A^2/3L^4 \qquad \phi = \theta_2 - \theta_1$$

A is the area of the vortex triangle, $L^2 = \sum |z_j|^2$, a constant of motion.

$$H = \frac{1}{2\pi} \sum_{i<j} \frac{1}{|z_j - z_i|} = f(I, \cos(3\phi)) \qquad \dot{I} = \frac{16I}{L^4} \frac{\partial H}{\partial \phi} = g(I, \sin(3\phi_1))$$

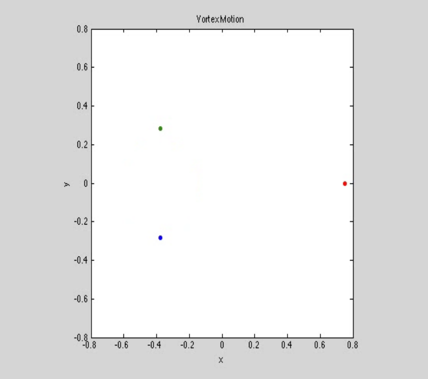
Using the system of equations, we numerically determine $\dot{I}(I)$ and find the potential function $-\dot{I}^2$.

Potential Well



$$T = 2\int_{I_{\min}}^{I_{\max}} \frac{dI}{|\dot{I}|}$$

Vortex Motion



First Order Expansion

We wish to determine the vertical velocity

$\epsilon = Ro$

$$\begin{aligned} u &\sim -\frac{\partial\psi_0}{\partial y} - \epsilon\left(\frac{\partial\psi_1}{\partial y} + \frac{\partial F_1}{\partial z}\right) \\ v &\sim \frac{\partial\psi_0}{\partial x} + \epsilon\left(\frac{\partial\psi_1}{\partial x} - \frac{\partial G_1}{\partial z}\right) \\ w &\sim \epsilon\left(\frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y}\right) = -\frac{D_0\theta_0}{Dt} \end{aligned}$$

F, G: associated potentials.

Muraki *et al.* (1999)

3D Transport

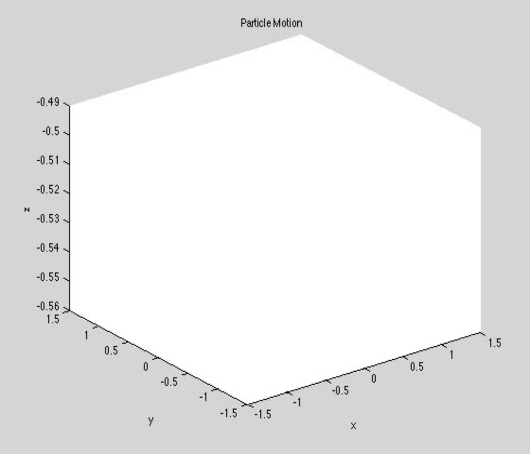
- For now take w_1 but not u_1, v_1:

$$w_1 = 3z \sum_n \frac{\kappa_n}{2\pi} \frac{\left(\vec{u}_0 - \dot{\vec{x}}_n\right) \cdot (\vec{x} - \vec{x}_n)}{|\vec{x} - \vec{x}_n|^5}$$

- We then compute Poincaré maps for particles advected by the flow and project the maps to the x-y plane.
- By comparing the plots created by particles at different heights, we observe the effect of 3D motion on the paths.

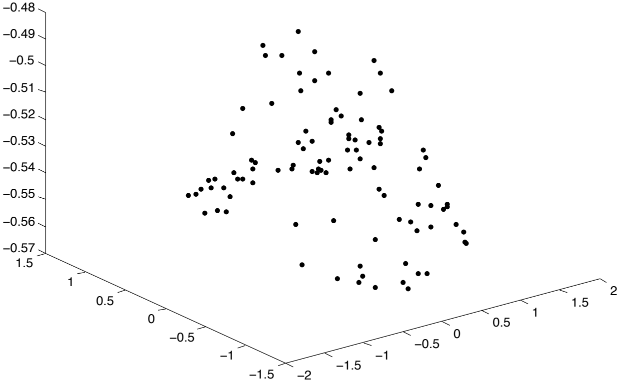
Particle Movie

$\epsilon = 0.1$



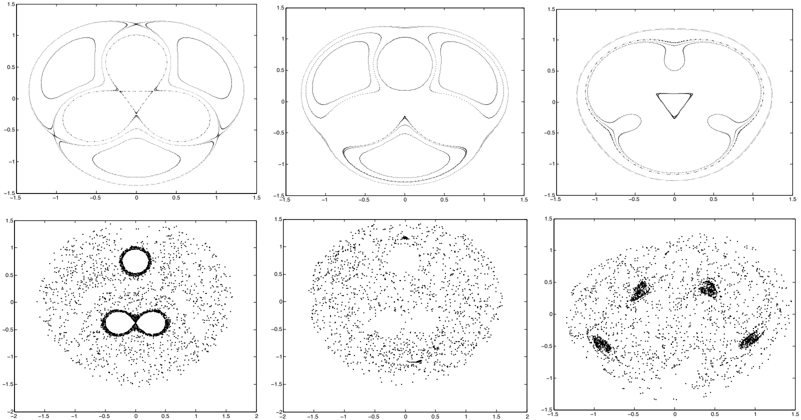
Poincaré Plot in 3D

$\epsilon = 0.1$



Poincaré 2D Projections at Various Heights

$\epsilon = 0.1$



$z = 0$ $z = -.25$ $z = -.5$

Conclusions and Future Work

- SQG vortices is a useful system for Ocean3D+1: 2D dynamics with 3D transport
- Fundamental question: at what time does the 3D model deviate from 2D? $O(Ro^{-1})$?
- Point vortex and other reduced systems provide a testbed for developing tools to look at 3D+1 transport.

$O(Ro)$ corrections

- Need to add u_1 and v_1 corrections. Technically lengthy.
- Using analytical flow fields extremely useful in developing approach for the more general case (multiple Poisson problems to solve).
- Interesting as pure GFD problem: linked to issues of slaving and slow manifolds.
- Very few explicit calculations have been carried out.

Finite depth

- Point vortices are in a sense self-similar. Add a lower surface to introduce a vertical scale.
- This can be viewed as a crude model of a mixed layer with a thermocline.
- Background flow leads to an interesting stability problem coupling barocliinc instability with surface ML instabilities. Could be used as dynamically constistent extension of classical LCS jet model?
- Can also add (weak) topography on one boundary.

Moment model

- Useful to obtain model with less singular velocity fields than point vortices.
- Typical way is to desingularize point vortices into e.g. vortex patches.
- Resulting contour dynamics is still complex. Simplify to SQG analog of 2D moment model.
- Transport properties of two patches using moment model have been investigated in 2D by Rizzi & Cortelezzi (2011). Examine in SQG.

2D moment transport picture

Collaborations

- Jones, Spiller: share dynamically consistent velocity fields for use in LA.
- Kirwan: compare 2D/3D FTLE calculations in SQG with model fields (ABC) and model output.
- Poje: examine vertical velocities for QG ellipsoids.

Work with Rodolphe Chabreyrie (now at GWU)

- Solomon and Mezić roll model. Two fast variables and one slow variables: KAM-like tori.
- Long-time behavior: widespread mixing
- Short-time behavior: alternation between horizontal and vertical mixing
- Similarities between 2D+1t and 2D+1t+1D

References

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