Dynamics of SQG Point Vortices and Passive Scalar Transport

Cecily Keppel
Stefan Llewellyn Smith

Mechanical and Aerospace Engineering
University of California, San Diego

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Ocean Dynamics

- The ocean is a thin fluid envelope spread on a rapidly rotating Earth. Dynamics are largely two-dimensional with weak vertical variation.
- Behavior is governed by the dimensionless Rossby number $Ro = U/Lf$, ratio of local vorticity to planetary vorticity.
- Goal: understand effect of weak 3D variation using dynamically consistent models

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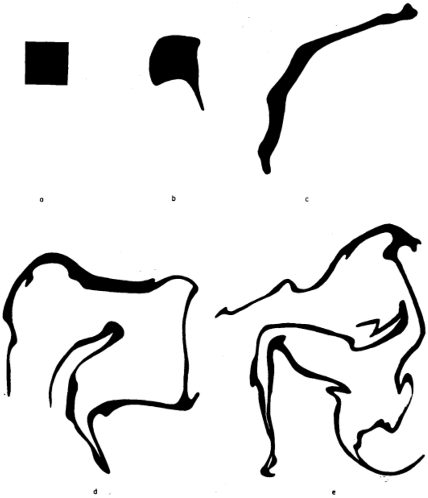
Stirring/Mixing (Welander, 1955)



Background

- In oceanography and fluid mechanics, people have developed tools to understand mixing based on Hamiltonian 2D dynamics (Samelson & Wiggins 2006, Mezic *et al* 2010)
- How are these tools relevant to properties of 3D flow? (Theme of MURI)
- The simplest possible 3D flow will be nearly 2D with weak variation in the third direction, consider ocean context.

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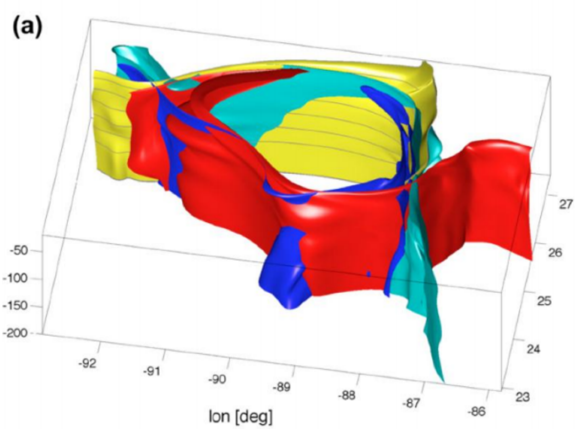
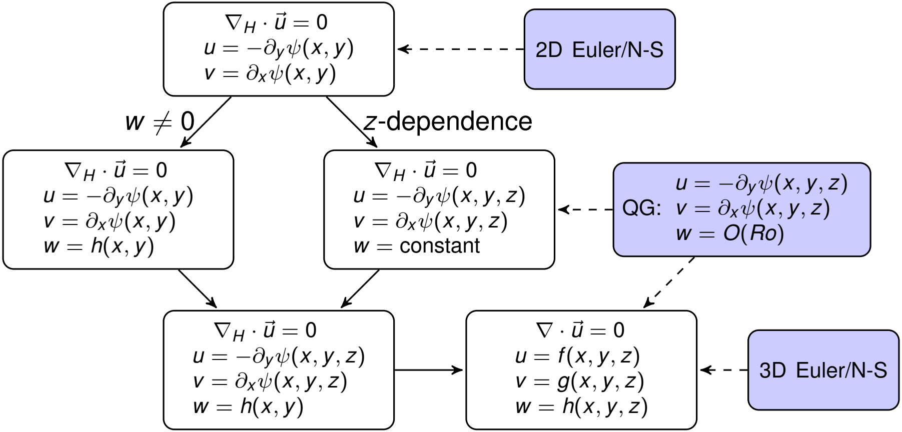
Gulf Stream Loop



Manifolds calculated numerically at 8 different heights, stitched together ad hoc.

Branicki and Kirwan 2010

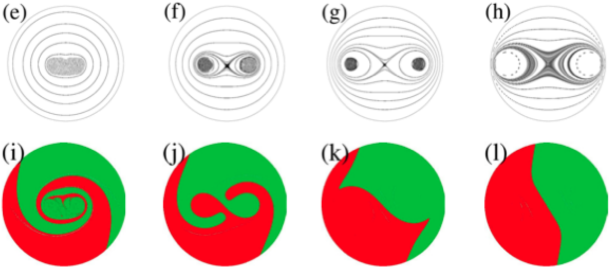
Can add t to any system



Dynamics ultimately come from rotating Boussinesq Navier-Stokes

Current Tools

- Lagrangian tools
  - FTLE
  - Mix-norm (Matthew *et al* 2005)
  - Lobe dynamics (Wiggins 2005)
- Eulerian tools
  - Eulerian map/indicators (Sturman & Wiggins 2008)
- Perturbative
  - KAM



Stirring vs. Mixing

- Previous tools are indicators of stirring
- Shear has a strong effect on mixing (e.g. Taylor dispersion)
- What is the effect of vertical shear on these systems?
- What tools take this ($Pé = UL/D$) into account?

The SQG Model

- QG and SQG are valid for $Ro \ll 1$
- They conserve potential vorticity
- SQG has 0 potential vorticity in the interior and some θ on the boundary (Held *et al* 1995)
- SQG is nice because the *dynamics are 2D* while the *interior fluid motion is 3D*!

Quasigeostrophic (QG) Equations

Reduced equation of motion for $Ro \ll 1$

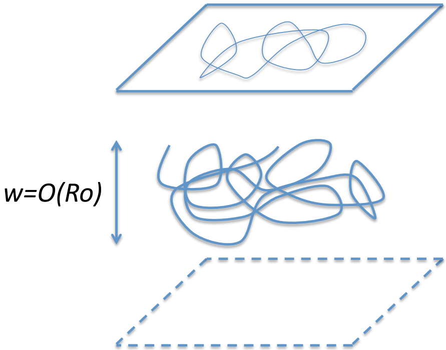
$$\begin{array}{rl} \zeta & \text{Vorticity} \\ \theta = f\partial_z\psi & \text{Buoyancy } (\sim \text{density}) \\ \psi & \text{Streamfunction} \end{array}$$

$$\begin{array}{rrcl} \text{bulk} & \partial_t\zeta & = & -J(\psi,\zeta) + f\partial_z w \\ \text{surface} & \partial_t\theta & = & -J(\psi,\theta) - N^2 w \\ \text{bulk} & q & = & \left[\partial_{xx} + \partial_{yy} + \partial_z\left(\frac{f}{N}\right)^2\partial_z\right]\psi \end{array}$$

Pedlosky 1982

Surface Quasigeostrophic Equations

Surface QG (SQG) assumes potential vorticity $q = 0$, so the only forcing comes from the boundary (such as the surface)



$\theta = f\frac{\partial\psi}{\partial z}$ evolves on the surface

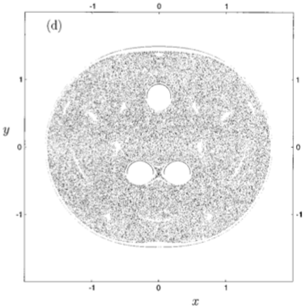
Motion induced below where q=0

Possible second layer?

3-Vortex Problem

We want the simplest possible dynamically consistent flow field:

Three classical point vortices (which have regular motion) induce chaotic motion in passive scalars. We wish to understand the transport properties of SQG point vortices in the fluid interior and boundary.



Kuznetsov and Zaslavsky 1998

Equations of Motion on the Surface

$$\theta_0 = \sum_n \kappa_n \delta(x - x_n)\delta(y - y_n)$$

$$(\dot{x}_n, \dot{y}_n) = \sum_{m \neq n} \frac{\kappa_m}{2\pi} \frac{1}{|\vec{x}_n - \vec{x}_m|^3}(-y_n + y_m, x_n - x_m)$$

Hamiltonian system:

$$\psi_0 = \frac{1}{2\pi|\vec{x} - \vec{x}_n|}$$

$$\vec{u}_0 = \sum_n \frac{\kappa_n}{2\pi|\vec{x} - \vec{x}_n|^3}(-y + y_n, x - x_n, 0)$$

Dynamically (asymptotically) consistent

Change of Variables

We follow the analysis for classical point vortices, all of unit strength, by Aref & Pomphrey (1982) and Kuznetsov & Zaslavsky (1998):

$$z_j = \frac{1}{\sqrt{3}} \sum_{n=1}^{2} \sqrt{2J_n} e^{i\theta_n} e^{-2i\pi n(j-1)/3}$$

$$I_1 = \frac{J_2 - J_1}{2} = A_{123}/\sqrt{3} \qquad I_2 = \frac{J_2 + J_1}{2} = L^2/4$$

A_{123} is the signed area of the vortex triangle, $L^2 = \sum |z_j|^2$

$$\phi_1 = \theta_2 - \theta_1 \qquad \phi_2 = \theta_2 + \theta_1$$

Change of Variables

Finally let $I = \left(\frac{I_1}{I_2}\right)^2 = \frac{16}{3} A^2 / L^4$.

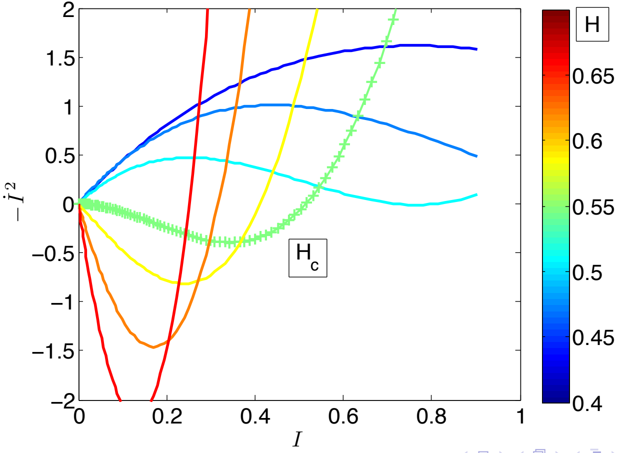
SQG: $$H = \frac{1}{2\pi} \sum_{i<j} \frac{1}{|z_j - z_i|}$$

Now $$H = f(I, \cos(3\phi_1))$$

Also $$\dot{I} = \frac{2I}{(I_2)^2} \frac{\partial H}{\partial \phi_1} = g(I, \sin(3\phi_1))$$

Using the system of equations, we numerically determine $\dot{I}(I)$ and consider the potential function $-\dot{I}^2$.

Potential Well

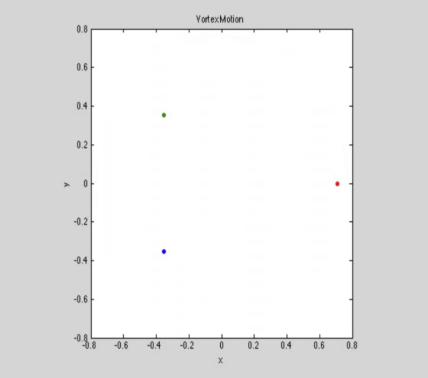


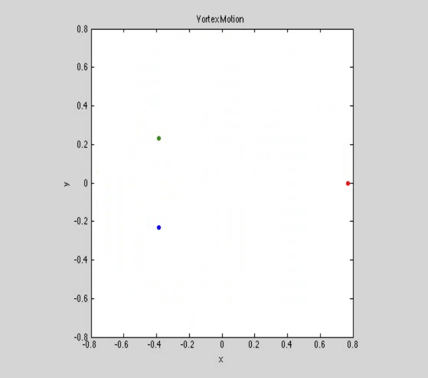
Calculating Period

$$T = 2\int_{I_{\min}}^{I_{\max}} \frac{dI}{|\dot{I}|}$$

Recall that $I \propto A$, so this period reflects a return to the initial area, but with vortices permuted.

In order to return to the initial state, we will need to multiply by either 2 or 3 depending on the regime of motion, yielding T_{rel}.


VortexMotion
y
x
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8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Calculating Shift

Here's where ϕ_2 matters...

Given $$\dot{\phi}_2 = -\frac{\partial H}{\partial I_2}$$

Then $$\phi_2 = \int^{I(t)} \frac{\dot{\phi}_2}{|\dot{I}|} dI$$

And the shift in angle is simply $\phi_2(T_{rel}) - \phi_2(0)$, found numerically from the above integral.

Interior Flow Field

We wish to determine the vertical velocity. In the asymptotic procedure to obtain QG equations, we begin by expanding the streamfunction and vorticity potentials ψ, F, G.

$\epsilon = Ro$

$$\begin{aligned}
\psi &\sim \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots \\
F &\sim \epsilon F_1 + \epsilon^2 F_2 + \cdots \\
G &\sim \epsilon G_1 + \epsilon^2 G_2 + \cdots
\end{aligned}$$

Muraki *et al* 1999

Interior Flow Field

Our system is governed by

$$\nabla^2 \psi_0 = 0 \qquad \frac{\partial \psi_0^s}{\partial z} = \theta_0$$

$$\nabla^2 F_1 = 2J\left(\frac{\partial \psi_0}{\partial z}, v_0\right) \qquad \nabla^2 G_1 = 2J\left(\frac{\partial \psi_0}{\partial z}, -u_0\right)$$

$$\left(\frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y}\right)^s = 0$$

$$\nabla^2 \psi_1 = \left|\nabla \frac{\partial \psi_0}{\partial z}\right|^2 \qquad \left(\frac{\partial \psi_1}{\partial_z} + \frac{\partial G_1}{\partial x} - \frac{\partial F_1}{\partial y}\right)^s = \theta_1^s = 0$$

where s indicates at the surface, $z = 0$.

Muraki *et al* 1999

First Order Expansion

Now we have

$$
\begin{aligned}
u &\sim -\frac{\partial \psi_0}{\partial y} - \epsilon\left(\frac{\partial \psi_1}{\partial y} + \frac{\partial F_1}{\partial z}\right) \\
v &\sim \frac{\partial \psi_0}{\partial x} + \epsilon\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial G_1}{\partial z}\right) \\
w &\sim \epsilon\left(\frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y}\right) = -\frac{D_0\theta_0}{Dt}
\end{aligned}
$$

F, G are potential vorticity

$\frac{D\theta_0}{Dt}$ known from surface evolution of θ_0

Muraki *et al* 1999

3D Transport

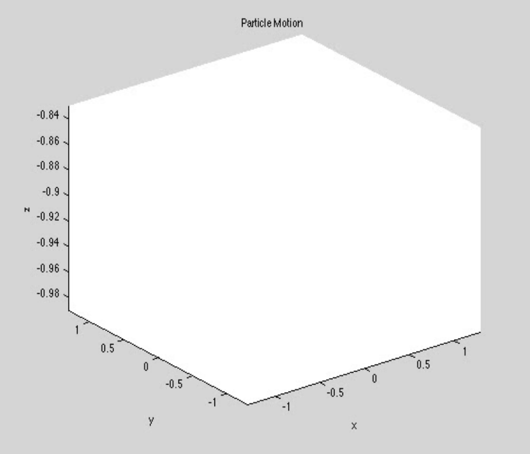
- For vortices we first take w_1 but not u_1, v_1 (yet unknown):

$$w_1 = 3z \sum_n \frac{\kappa_n}{2\pi} \frac{\left(\vec{u}_0 - \dot{\vec{x}}_n\right) \cdot (\vec{x} - \vec{x}_n)}{|\vec{x} - \vec{x}_n|^5}$$

- We then compute Poincaré maps for particles advected by the flow, and project the maps to the x-y plane.
- By comparing the plots created by particles at different heights, we observe the effect of 3D motion on the paths.

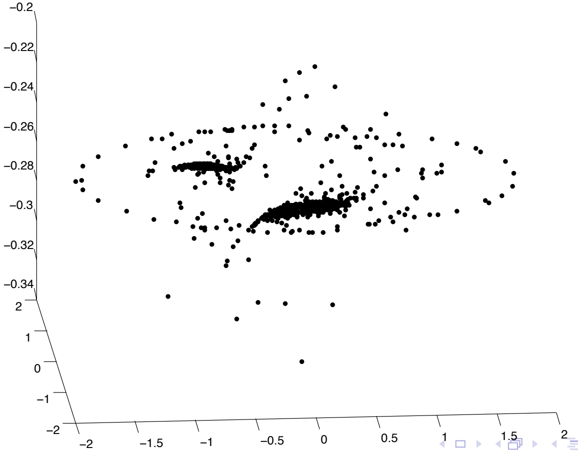
Particle Movie

$\epsilon = 0.1$



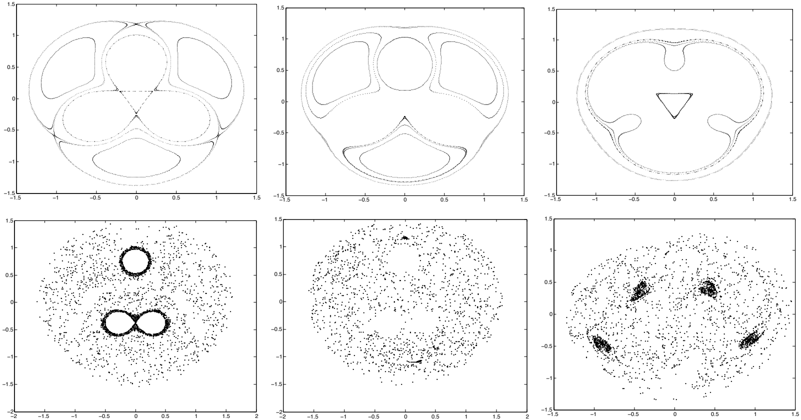
Poincaré Plot in 3D

$\epsilon = 0.1$



Poincaré 2D Projections at Various Heights

$\epsilon = 0.1$



$z = 0$ $z = -.25$ $z = -.5$

Conclusions

- SQG vortices constitute an interesting model system: 2D dynamics and 3D transport.
- Vortex motion can be described using geometric variables to numerically determine period and shift.
- Vortex motion is regular but transport can be chaotic.
- Particle's initial height has clear effect on transport.
- This system comes from "perturbing" Hamiltonian equations. Is there a connection to KAM which perturbs the Hamiltonian? Are there theorems that apply?

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Details to Examine

- Will u_1 and v_1 corrections further change the transport properties?
- At what time do the transport properties of the 3D model deviate from 2D? $O(1/Ro)$? (recall Mohamed Sulman's talk last year looking at dependence on strain rate)
- What if we add a second boundary, providing a new length scale?
- Will Péclet number have an effect via shear dispersion?
- What if we consider full SQG turbulence?
- What are the best tools for studying mixing in nearly 2D? In full 3D? How can we quantify errors of the tools?

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