SQG Vortex Dynamics and Passive Scalar Transport

Cecily Taylor
Stefan Llewellyn Smith

Mechanical and Aerospace Engineering
University of California, San Diego

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Ocean Dynamics

- The ocean is a thin fluid envelope spread on a rapidly rotating Earth. Dynamics are largely two-dimensional with weak vertical flow.
- Vortices are common in the ocean.
- Goal: understand effect of weak 3D variation using dynamically consistent models with weak vertical velocity
- Use idealized vortices as model problems.

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Quasigeostrophic (QG) Equations

Boussinesq, hydrostatic. Constant f, N.
Reduced equation of motion for $Ro \ll 1$.

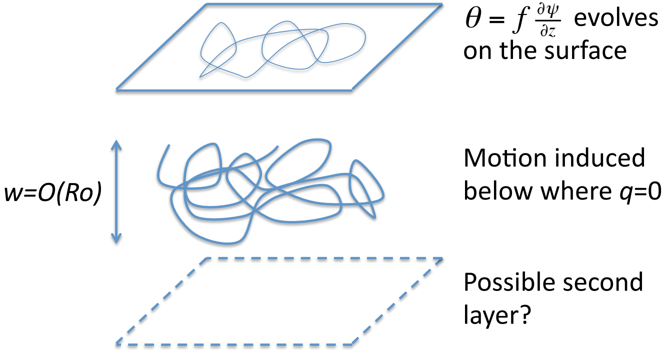
ζ	Vorticity
$\theta = f\partial_z\psi$	Buoyancy ($\sim$ density)
ψ	Streamfunction

bulk	$\partial_t\zeta = -J(\psi,\zeta) + f\partial_z w$
surface	$\partial_t\theta = -J(\psi,\theta) - N^2 w$
bulk	$q = \left[\partial_{xx} + \partial_{yy} + \left(\frac{f}{N}\right)^2 \partial_{zz}\right]\psi$

Pedlosky 1982

Surface Quasigeostrophic Equations

Surface QG (SQG) assumes potential vorticity $q = 0$, so the inherent dynamics are on the boundary



Interior Flow Field

We wish to determine the consistent velocity solution to $O(Ro)$. In the asymptotic procedure to obtain QG equations, begin by defining and expanding potentials ψ, F, G.

$$\begin{pmatrix} v \\ -u \\ \theta \end{pmatrix} = \nabla\psi + \nabla \times \begin{pmatrix} F \\ G \\ 0 \end{pmatrix}$$

$\epsilon = Ro$

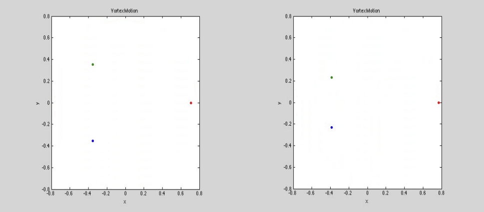
$$\begin{aligned} \psi &\sim \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots \\ F &\sim \epsilon F_1 + \epsilon^2 F_2 + \cdots \\ G &\sim \epsilon G_1 + \epsilon^2 G_2 + \cdots \end{aligned}$$

Muraki *et al.* 1999

Point Vortices

We want the simplest possible dynamically consistent flow field:

Three point vortices have regular motion but induce chaotic flow. Followed analysis by Kuznetsov & Zaslavsky (1998) for interaction of three vortices of equal strength.



SQG Point Vortex at $O(1)$

Singular vorticity distribution at $(x_n, y_n, 0)$

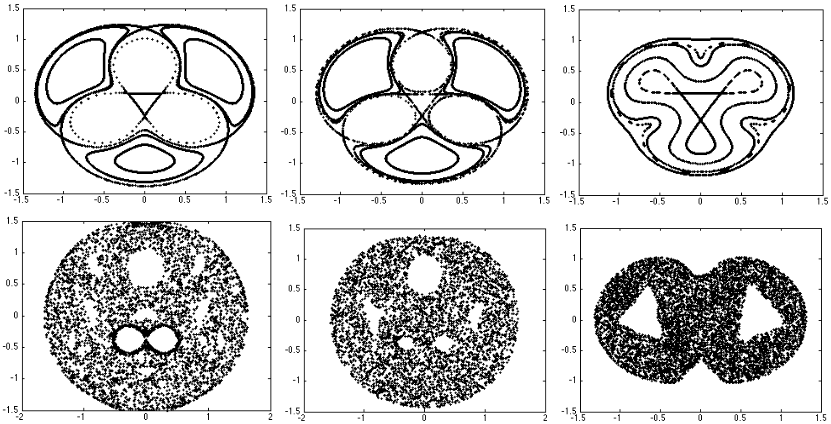
$$\theta_0^s = \sum_n \kappa_n \delta(x - x_n)\delta(y - y_n)$$

$$\nabla^2 \psi_0 = 0 \qquad \frac{\partial \psi_0^s}{\partial z} = \theta_0^s$$

$$\psi_0 = \sum_n \frac{\kappa_n}{2\pi|\vec{x} - \vec{x}_n|}$$

Dynamically (asymptotically) consistent

Poincaré Maps at Various Heights



$z = 0$ $z = -.25$ $z = -.5$

Arbitrary Strength

Following Aref (1979) use momentum conservation to define a constant C and trilinear coordinates

$$\kappa_1\kappa_2 l_{12}^2 + \kappa_2\kappa_3 l_{23}^2 + \kappa_3\kappa_1 l_{31}^2 = 3\kappa_1\kappa_2\kappa_3 C$$

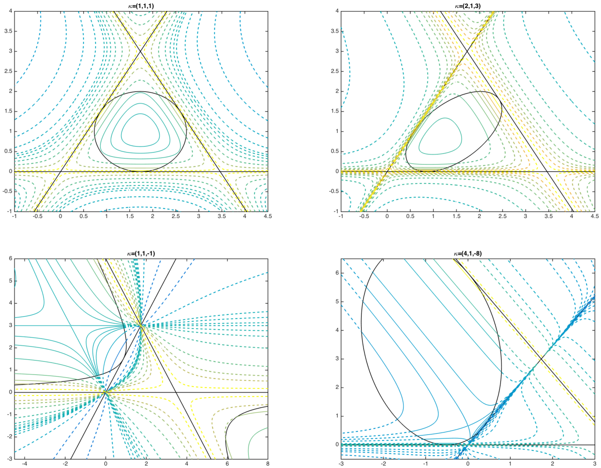
$$b_1 = \frac{l_{23}^2}{\kappa_1 C}, \qquad b_2 = \frac{l_{13}^2}{\kappa_2 C}, \qquad b_3 = \frac{l_{12}^2}{\kappa_3 C}$$

From

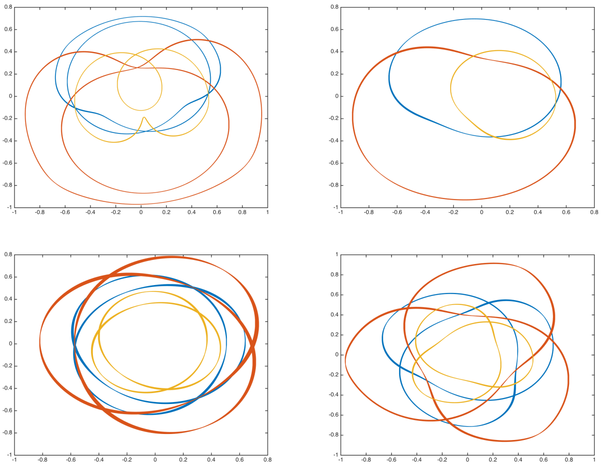
$$H = -\frac{1}{4\pi}\sum_{\alpha,\beta}\frac{\kappa_\alpha\kappa_\beta}{l_{\alpha\beta}}$$

$$= -\frac{\kappa_1\kappa_2\kappa_3}{2\pi|C|^{1/2}}\left(\frac{1}{|b_1\kappa_1|^{1/2}\kappa_1} + \frac{1}{|b_2\kappa_2|^{1/2}\kappa_2} + \frac{1}{|b_3\kappa_3|^{1/2}\kappa_3}\right)$$

Trajectory Curves



$\kappa = (2, 1, 3)$



$O(Ro)$ Corrections

The point vortices prove problematic when we attempt to find the $O(Ro)$ velocities.

Consider

$$\nabla^2 F_1 = 2J\left(\frac{\partial \psi_0}{\partial z}, \frac{\partial \psi_0}{\partial x}\right)$$

$$\psi_0 = \sum_n \frac{\kappa_n}{2\pi|\vec{x} - \vec{x}_n|}$$

So the equation of motion of F_1 involves multiplying derivatives of a singular function!

Can only calculate $\quad w = -\dfrac{D_0\theta_0}{Dt}$

Topological Entropy

Let (X, d) be a compact metric space with distance function d, and let $f\colon X \to X$ be a continuous self-map of X. Let $\varepsilon > 0$ be a positive real number, and let n be a positive integer. An *n-orbit* is a sequence x, $f(x), \dots, f^{n-1}(x)$ of f-iterates of a point x in X. Two n-orbits $\{f^i x\}$, $\{f^i y\}$, $0 \leqslant i < n$, are *ε-distinguishable* if there is a $j \in [0, n)$ for which $d(f^j x, f^j y) > \varepsilon$. Let $r(n, \varepsilon, f)$ denote the maximal number of ε-distinguishable n-orbits. It is easy to see that there are numbers $C > 0$ and $\alpha > 0$ such that $r(n, \varepsilon, f) \leqslant Ce^{n\alpha}$ for $n \geqslant 0$.

Let

$$r(\varepsilon, f) = \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon, f)$$

and let

$$h(f) = \lim_{\varepsilon \to 0} r(\varepsilon, f)$$

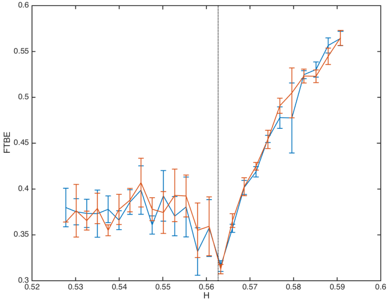
The number $h(f)$ is the *topological entropy* of f. For ε small, f has roughly $e^{nh(f)}$ ε-distinguishable n-orbits.

Topological Entropy

- Literature is dense: based on mapping analysis
- A single number for a given flow that doesn't involve a tracer advection equation
- Braiding Entropy - code from Jean-Luc Thiffeault and Marko Budisic
  - Built for 2D transport, but works for 3D trajectories (where z component is ignored) - is info lost?
  - Well commented and documented, no additional parameters

Initial Findings: FTBE vs. H

$\kappa = (1, 1, 1)$



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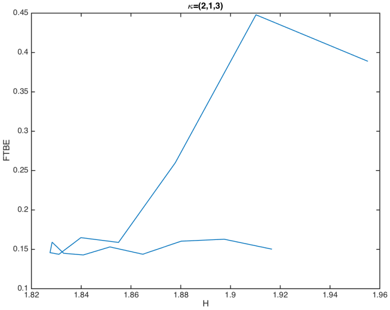
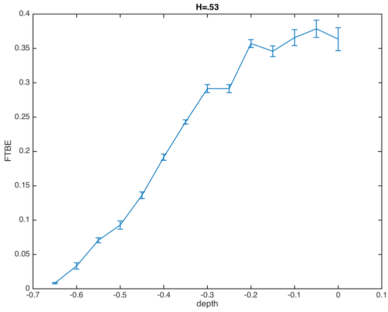
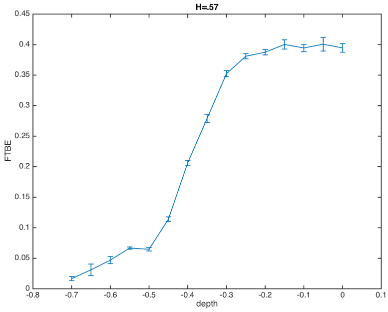


Initial Findings: FTBE vs. depth

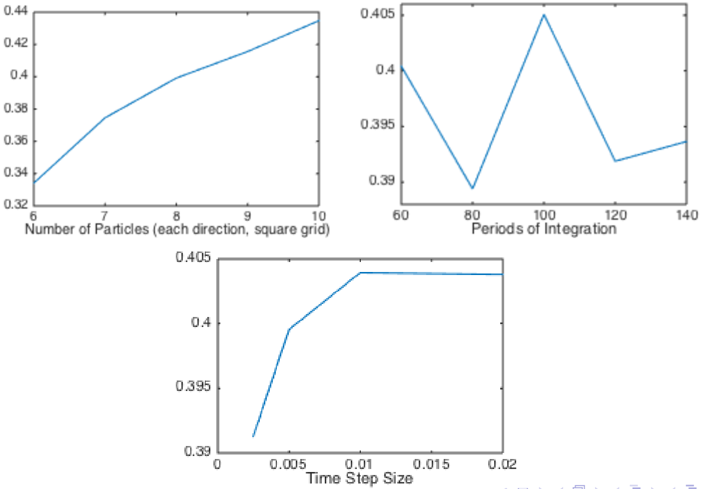
$H = .53$



$H = .57$

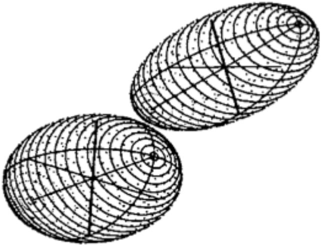


FTBE Variance



Dritschel Vortex

Dritschel (2011) described vortices that maintain their shape in SQG. Start with an ellipsoid oriented along Cartesian axes that contains a region of constant potential vorticity *Q*. In QG flow, this ellipsoid will rotate as a rigid body.



Dritschel *et al.* 2004

Limit to SQG

Take the limit as the vertical axis of the ellipsoid $c \to 0$. We find a buoyancy distribution

$$\theta_0^s = \kappa\sqrt{1 - x^2/a^2 - y^2/b^2}$$

where κ is related to vortex strength.

This is *continuous* so the flow field is now finite and regular, even at the edge.

Streamfunction

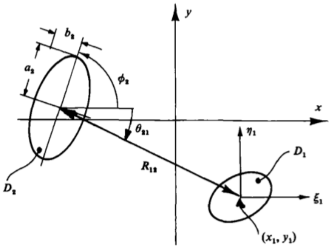
The streamfunction is then calculated from $\nabla^2\psi = 0$ and $\psi_z^s = \theta_0^s$ (Dritschel *et al.* 2004)

$$\psi = \frac{3\kappa}{4}\int_\lambda^\infty \frac{du}{\sqrt{(u+a^2)(u+b^2)u}}\left(1 - \frac{x^2}{u+a^2} - \frac{y^2}{u+b^2} - \frac{z^2}{u}\right)$$

λ is the largest root of $$\frac{x^2}{\lambda+a^2} + \frac{y^2}{\lambda+b^2} + \frac{z^2}{\lambda} = 1$$

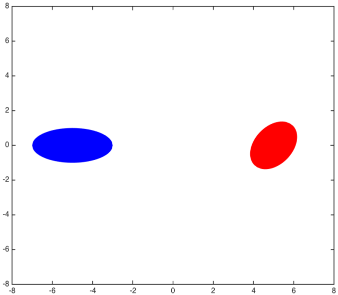
Vortex Interactions

Each vortex is represented by a set of point vortices, the strengths and positions of which are chosen to match the spatial moments of the initial elliptical vortex up to the desired order (Dritschel *et al.* 2004)



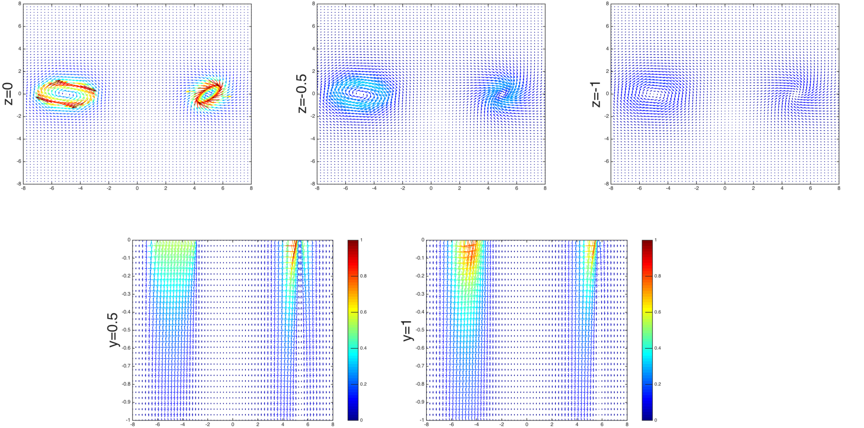
Simulation

$$a_1/b_1|_{t=0} = 2, \quad a_2/b_2|_{t=0} = 1.5, \quad R|_{t=0} = 10, \quad \kappa_1 = \kappa_2 = 1$$



Velocities: X-Y & X-Z slices

$\epsilon = 0.1$



Conclusions

- SQG vortices constitute an interesting model system: 2D dynamics and 3D transport.
- Vortex motion is regular but transport can be chaotic. Arbitrary vortex strengths introduce additional free parameters.
- Is it possible to reconcile singularities of point vortices to get $O(Ro)$ solution?

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Next Steps

- Run diagnostic for Dritschel vortex solution & interactions – use steady state solutions from Dritschel & Poje.
- Include $O(Ro)$ corrections and examine mixing dependence on Ro.
- At what time do the transport properties of the 3D model deviate from 2D? $O(1/Ro)$?
- What about a periodic domain? What if we add a second boundary, providing a vertical length scale?

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Interior Flow Field

Our system is governed by

$$\nabla^2 \psi_0 = 0 \qquad \frac{\partial \psi_0^s}{\partial z} = \theta_0^s$$

$$\nabla^2 F_1 = 2J\left(\frac{\partial \psi_0}{\partial z}, v_0\right) \qquad \nabla^2 G_1 = 2J\left(\frac{\partial \psi_0}{\partial z}, -u_0\right)$$

$$(F_1)^s = (G_1)^s = 0$$

$$\nabla^2 \psi_1 = q_1 + \left|\nabla \frac{\partial \psi_0}{\partial z}\right|^2 \qquad \left(\frac{\partial \psi_1}{\partial z}\right)^s = \theta_1^s = 0$$

where s indicates at the surface, $z = 0$.

Muraki *et al.* 1999

Solving Poisson Equations

In 2002, Hakim *et al.* determined the particular solutions to the QG^{+1} potentials. The resulting Laplace equation can be solved in horizontal 2D Fourier Space. For example:

particular solution to Poisson
$$F_1 = \frac{\partial \psi_0}{\partial y}\frac{\partial \psi_0}{\partial z} + \tilde{F}_1 \ ,$$

new gov'ing equations
$$\nabla^2 \tilde{F}_1 = 0, \qquad \tilde{F}_1{}^s = -\left[\frac{\partial \psi_0}{\partial y}\frac{\partial \psi_0}{\partial z}\right]^s \ ,$$

solution to Laplace
$$\hat{\tilde{F}}_1 = \hat{\tilde{F}}_1^s e^{\kappa z}$$

And similarly for G_1, ψ_1

Procedure

1. Specify θ_0^s with $\theta_1^s = 0$
2. Solve $\nabla^2 \psi_0 = 0$ with $\frac{\partial \psi_0}{\partial z} = \theta_0$ at $z = 0$
3. Solve Laplace equations for F^1, G^1, ψ^1.
4. Obtain velocities from derivatives of potentials.
5. Advect one time step and iterate.

Velocities

Now we have

$$\begin{aligned} u &\sim -\frac{\partial \psi_0}{\partial y} - \epsilon\left(\frac{\partial \psi_1}{\partial y} + \frac{\partial F_1}{\partial z}\right) \\ v &\sim \frac{\partial \psi_0}{\partial x} + \epsilon\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial G_1}{\partial z}\right) \\ w &\sim \epsilon\left(\frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y}\right) = -\frac{D_0\theta_0}{Dt} \end{aligned}$$

$$\frac{D\theta_0}{Dt} \text{ known from } \theta_0 = \frac{\partial \psi_0}{\partial z}$$

Muraki *et al.* 1999