RESONANCE PHENOMENA IN 3D TIME-DEPENDENT EKMAN - DRIVEN EDDY

IRINA RYPINA, LARRY PRATT, PENG WANG, TAMAY ÖZGÖKMEN, AND IGOR MEZIC

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Outline:

- Rotating can flow
- Steady symmetric background
- Steady symmetry-breaking disturbance
  - chaotic stirring over parameter ranges appropriate for ocean eddies
- Unsteady disturbance
  - barriers, manifolds, resonances
  - weakly nonlinear resonance analysis
  - Geometry of the flow near resonances
- Summary and conclusions

Dimensionless parameters:

$$Ro = |\delta\Omega| / \Omega$$

$$E = \nu / \Omega H^2$$

$$\mathrm{Re} = UR/\nu \text{ (related to } Ro \text{ and } E)$$

For ocean eddies

$$10^{-4} \le E \le 1$$

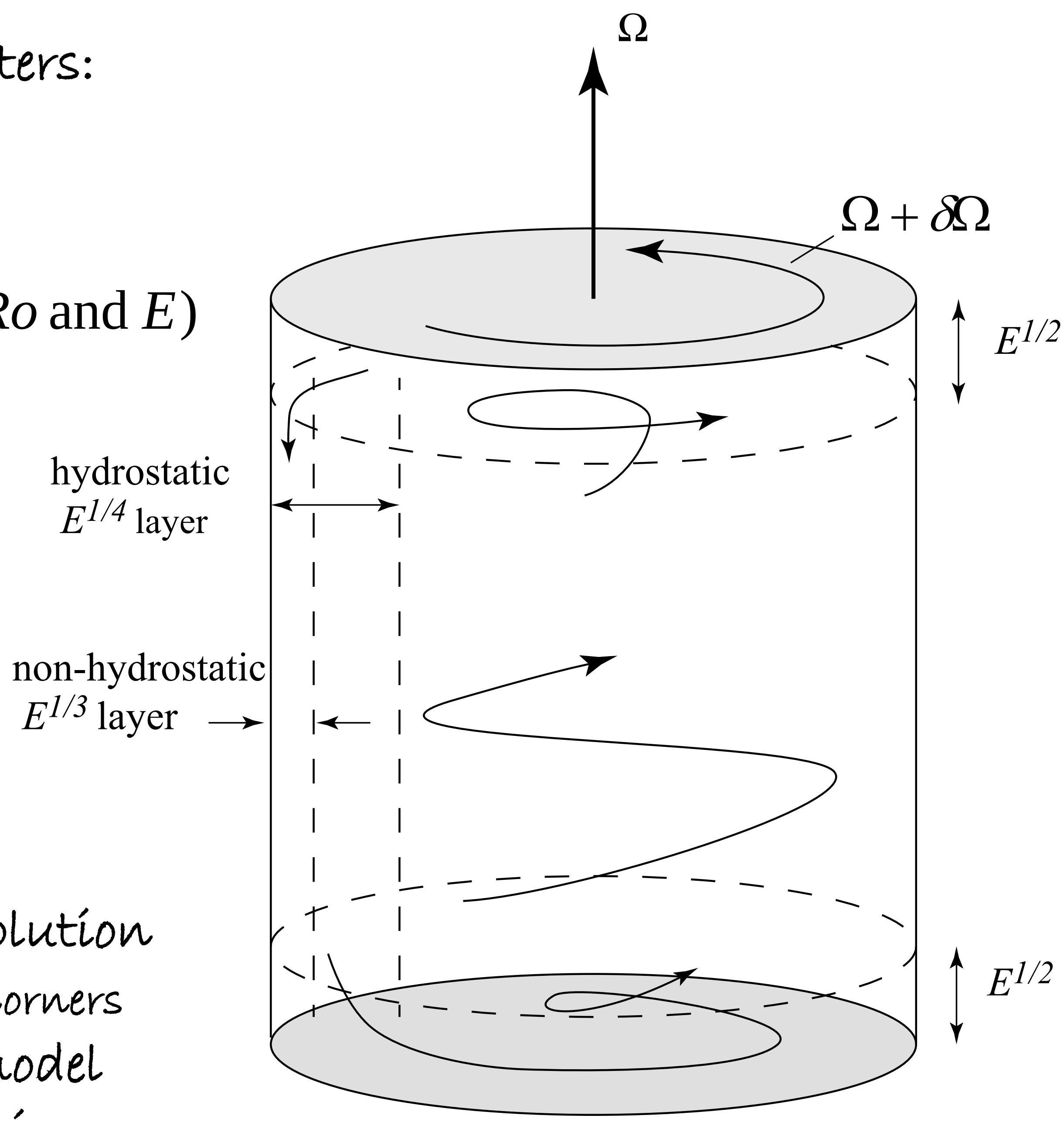
$$0.2 \le Ro \le 1$$

$$(0.2 \le \mathrm{Re} \le 2000)$$

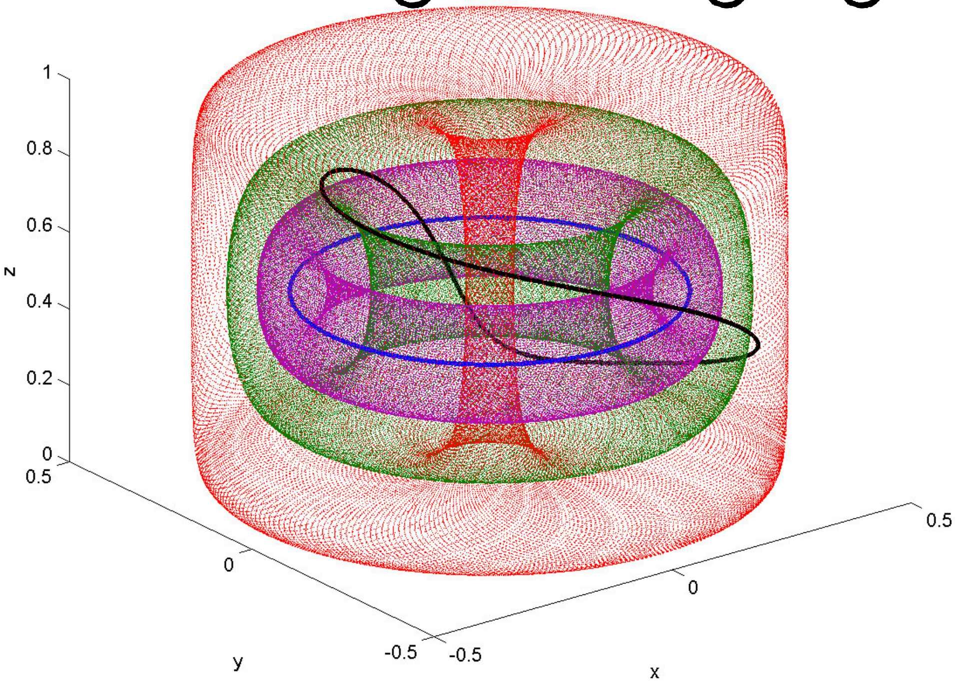
✓ Linear analytical solution exists but has problems in corners

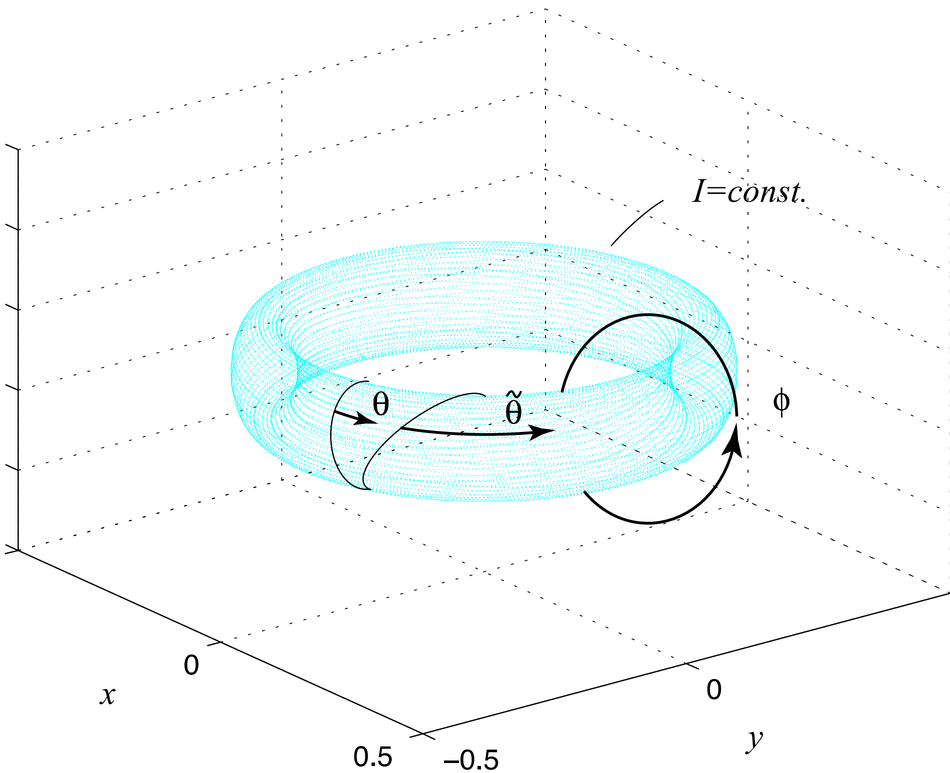
✓ Phenomenological model

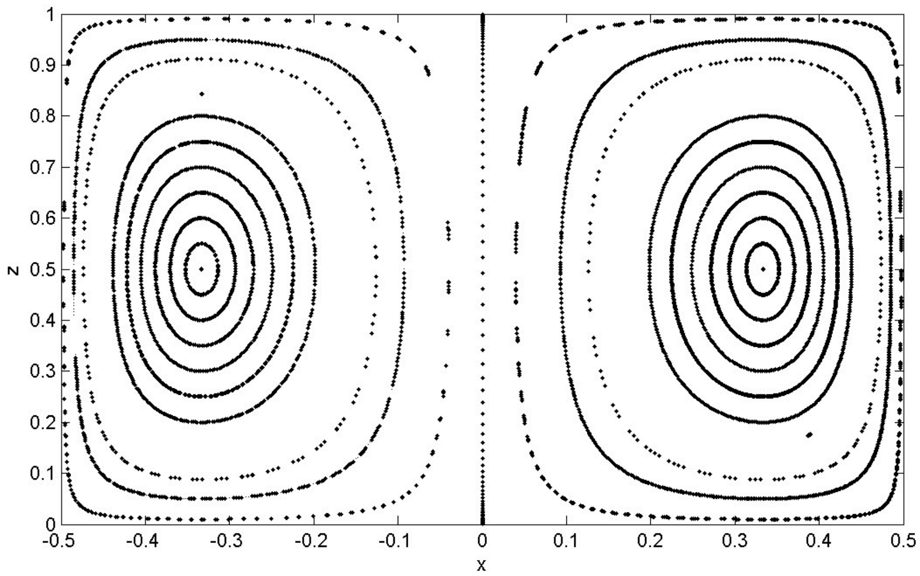
✓ Full numerical solution



Steady axially-symmetric background





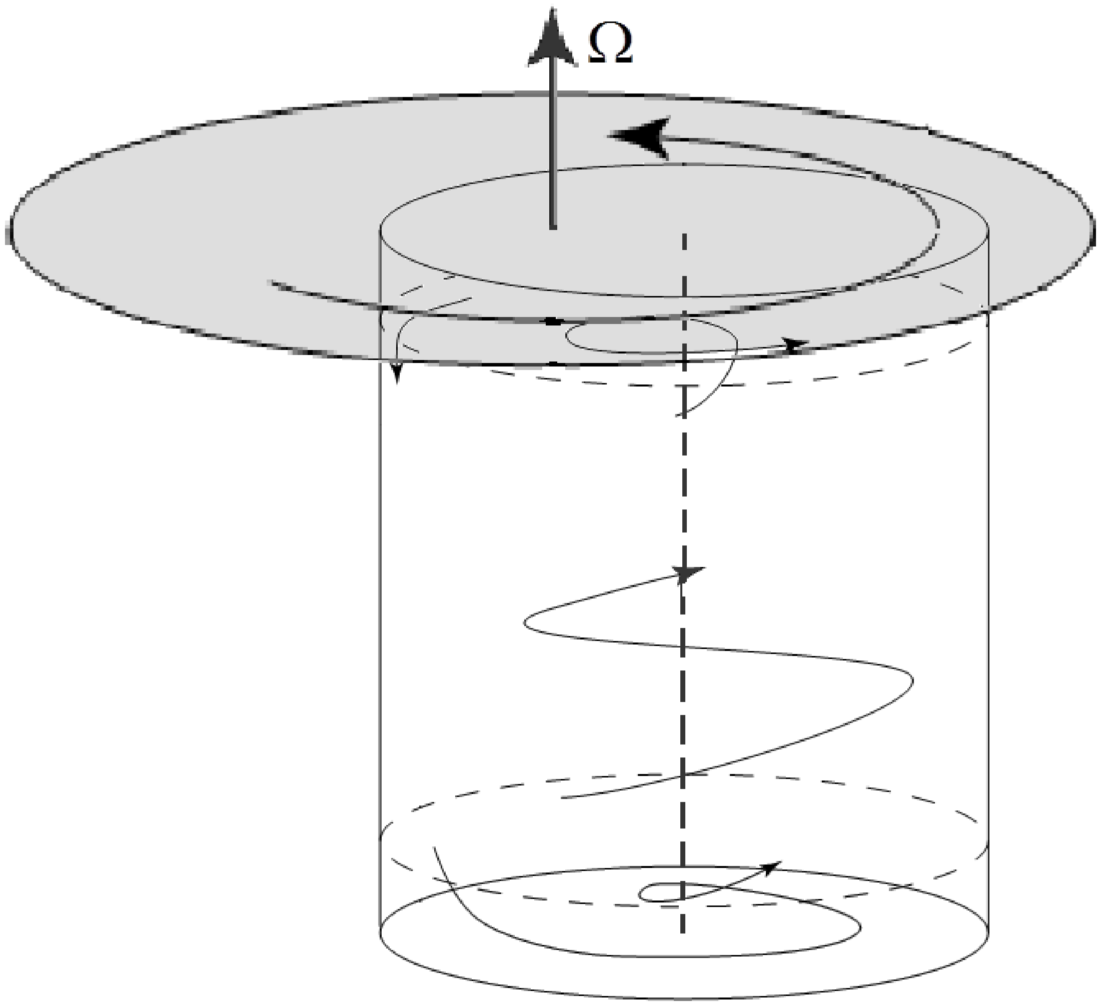


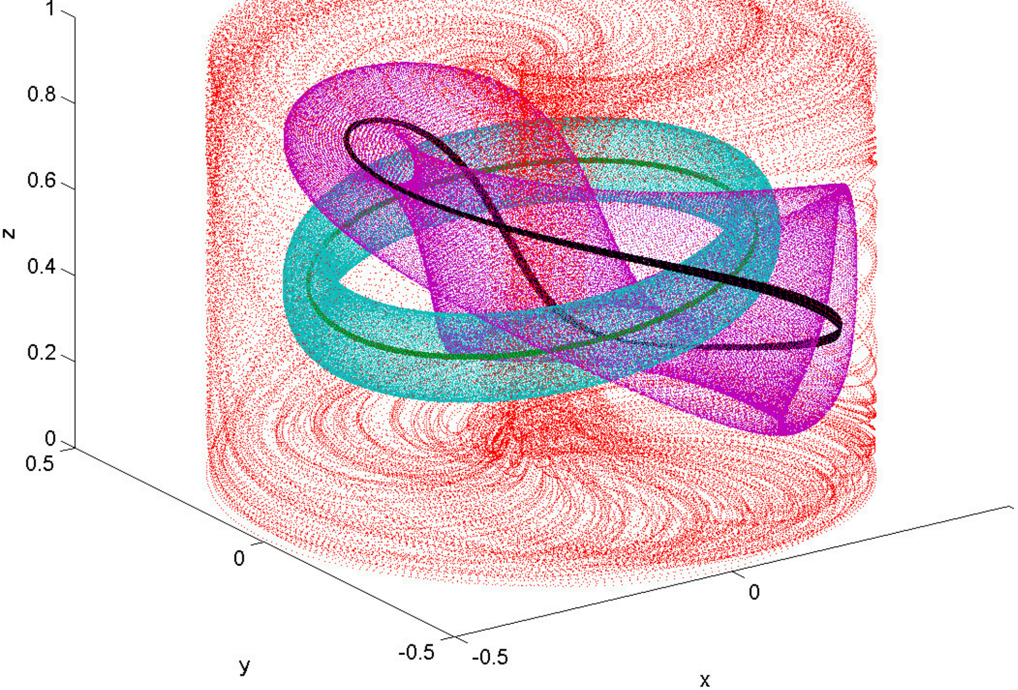
$$\dot{I} = 0$$

$$\dot{\phi} = \Omega_{\phi}(I)$$

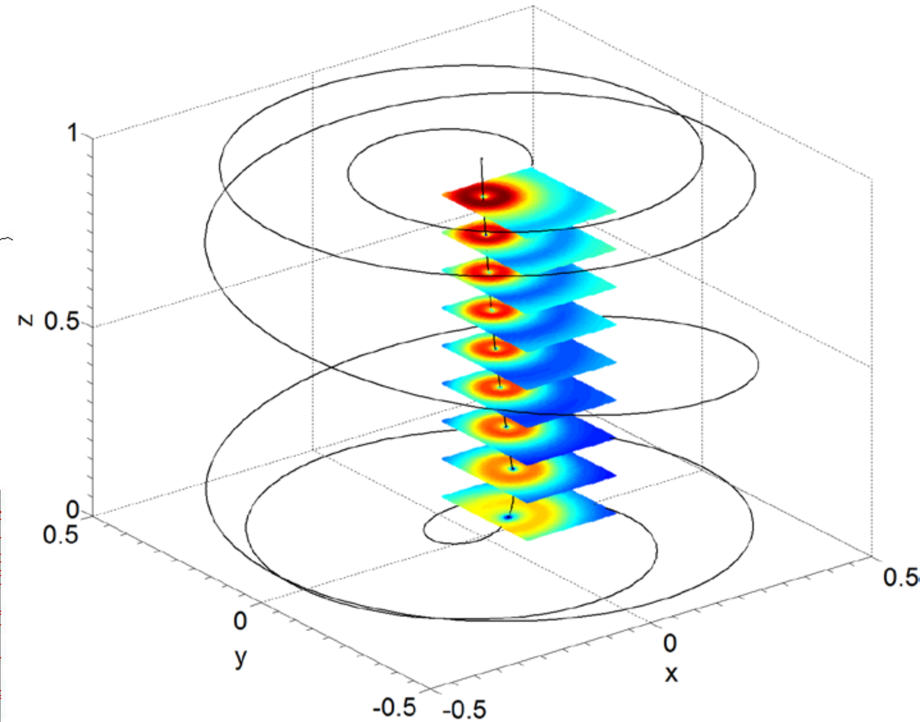
$$\dot{\tilde{\theta}} = \Omega_{\theta}(I)$$

Steady perturbation

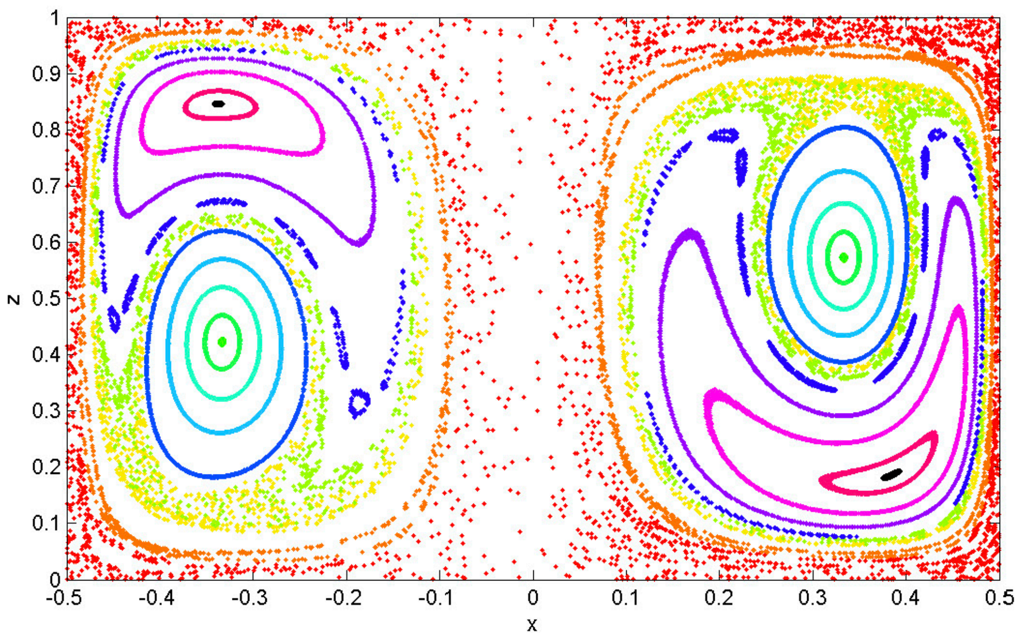




1
0.8
0.6
z
0.4
0.2
0
0.5
0
y
-0.5
-0.5
0
x



1
z
0.5
0
0.5
0
y
-0.5
-0.5
0
x
0.5

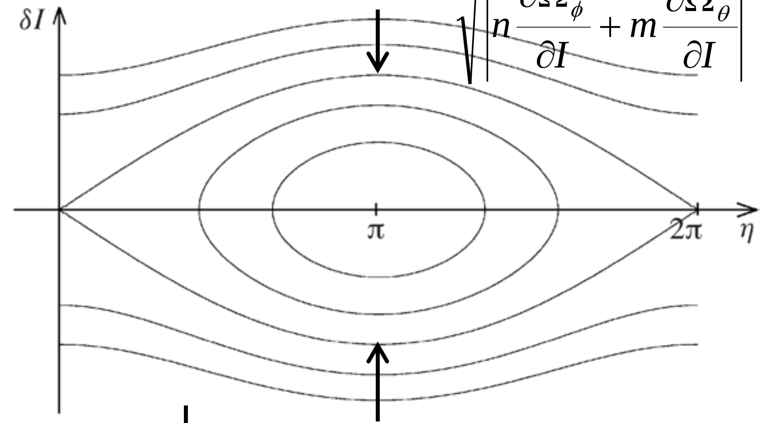


1
0.9
0.8
0.7
0.6
z
0.5
0.4
0.3
0.2
0.1
0
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
x

Analysis of the flow near resonances

$$\Delta I = 2\sqrt{\frac{\varepsilon F_{nm}(I_0)}{\left|n\frac{\partial\Omega_\phi}{\partial I}+m\frac{\partial\Omega_\theta}{\partial I}\right|}}$$

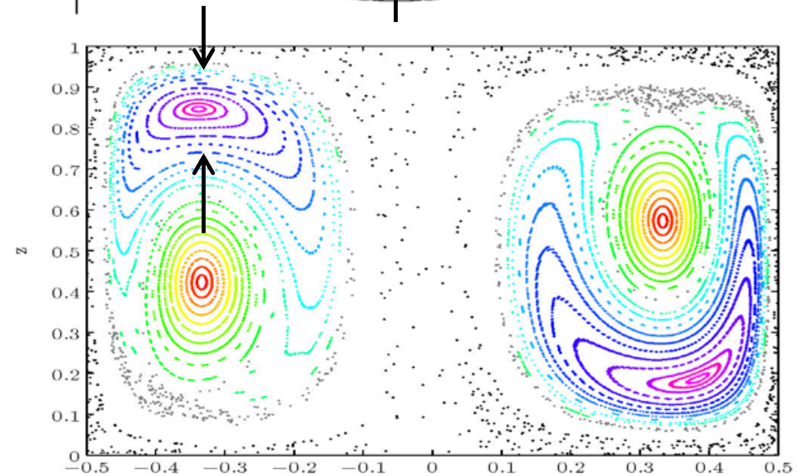
$$\begin{cases}\dot{I} = \varepsilon F^0(I,\phi,\theta)\\ \dot{\phi} = \Omega_\phi(I)+\varepsilon F^1(I,\phi,\theta)\\ \dot{\theta} = \Omega_\theta(I)+\varepsilon F^2(I,\phi,\theta)\end{cases}$$



$$\dot{I} = \varepsilon\,\Sigma F_{nm}(I)\sin(n\phi+m\theta+\alpha_{nml})$$

$$\eta = n\phi+m\theta+\alpha_{nm} =$$

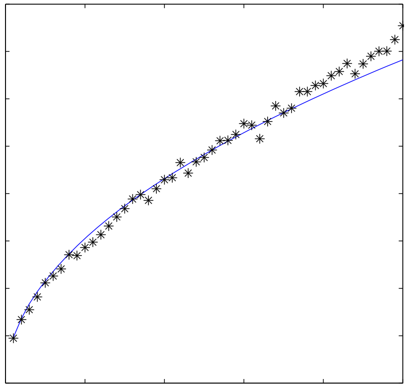
$$n\phi_0+m\theta_0+\alpha_{nm}+t(n\Omega_\phi+m\Omega_\theta)+O(\delta I)$$



Main contribution is from

$$n\Omega_\phi(I_0)+m\Omega_\theta(I_0)=0$$

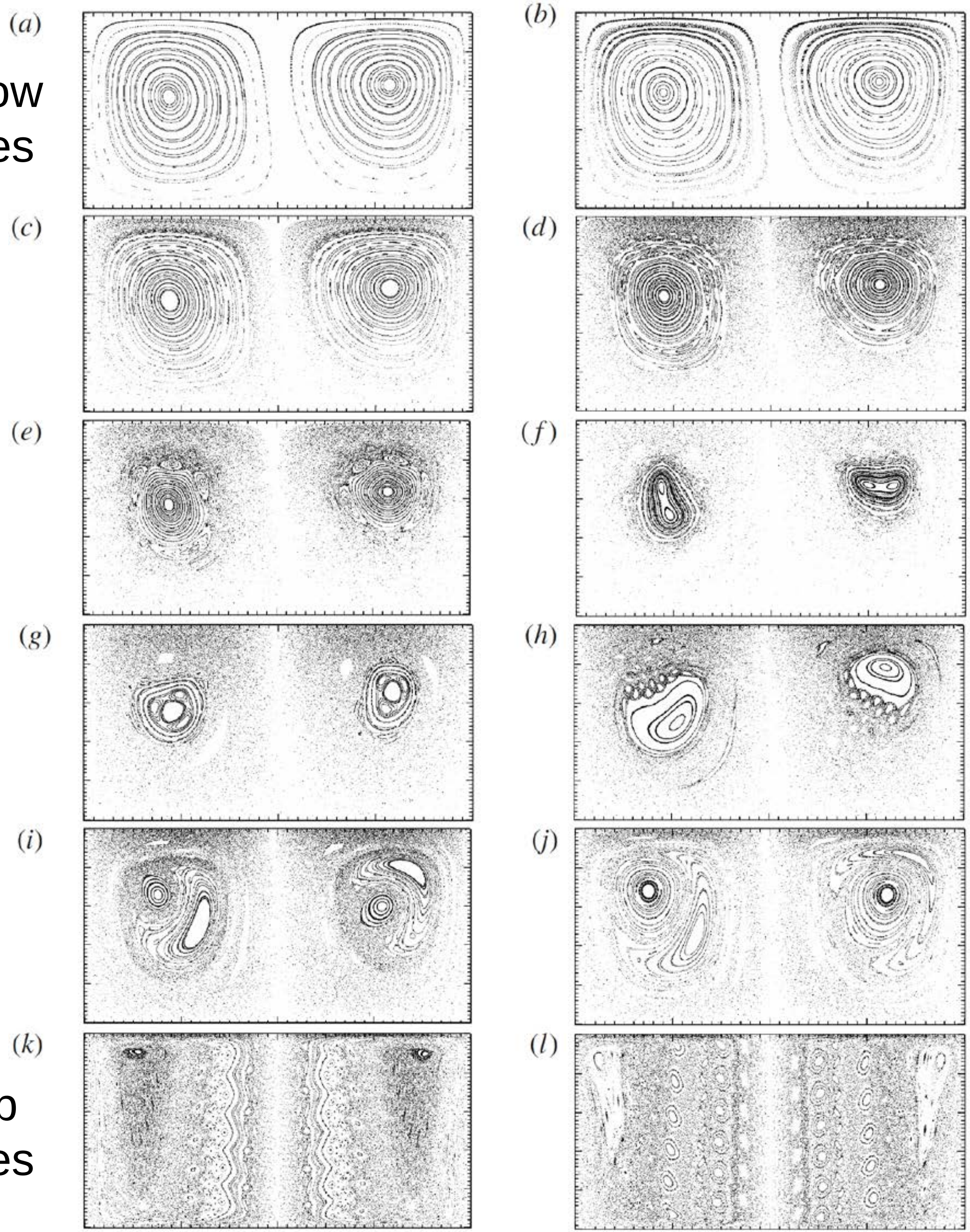
Linearized system

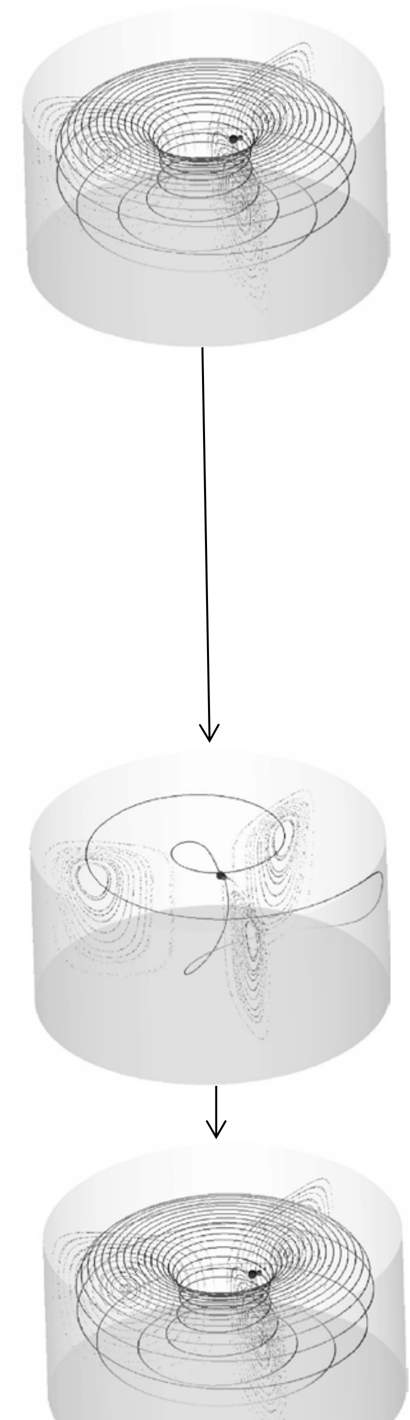


$$\begin{cases}\delta\dot{I} = -\partial H/\partial\eta\\ \dot{\eta} = \partial H/\partial\eta\end{cases}$$

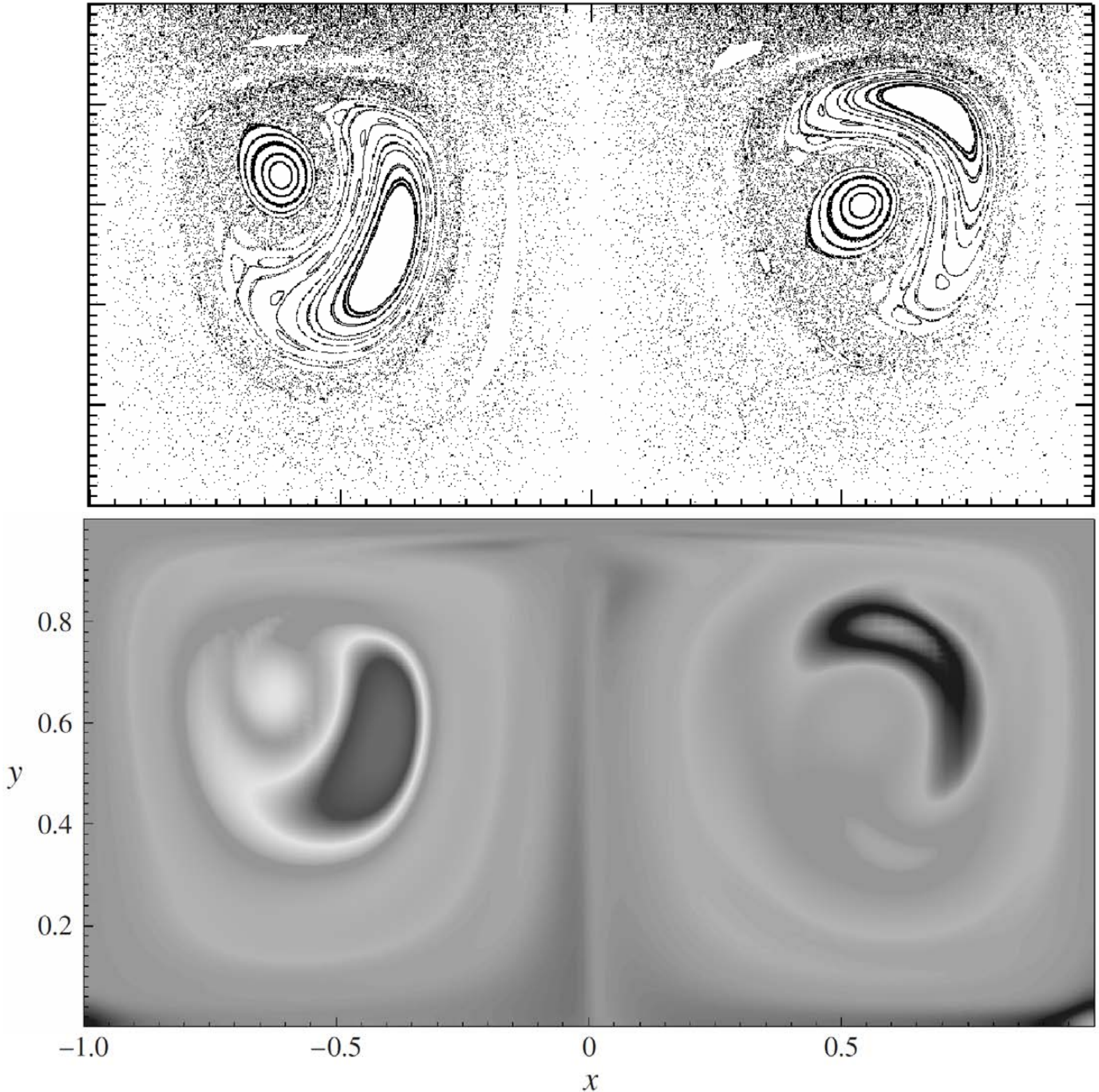
$$H = \varepsilon F_{nm}\cos\eta+\left(n\frac{\partial\Omega_\phi}{\partial I}+m\frac{\partial\Omega_\theta}{\partial I}\right)\frac{\delta I^2}{2}$$

shallow
eddies

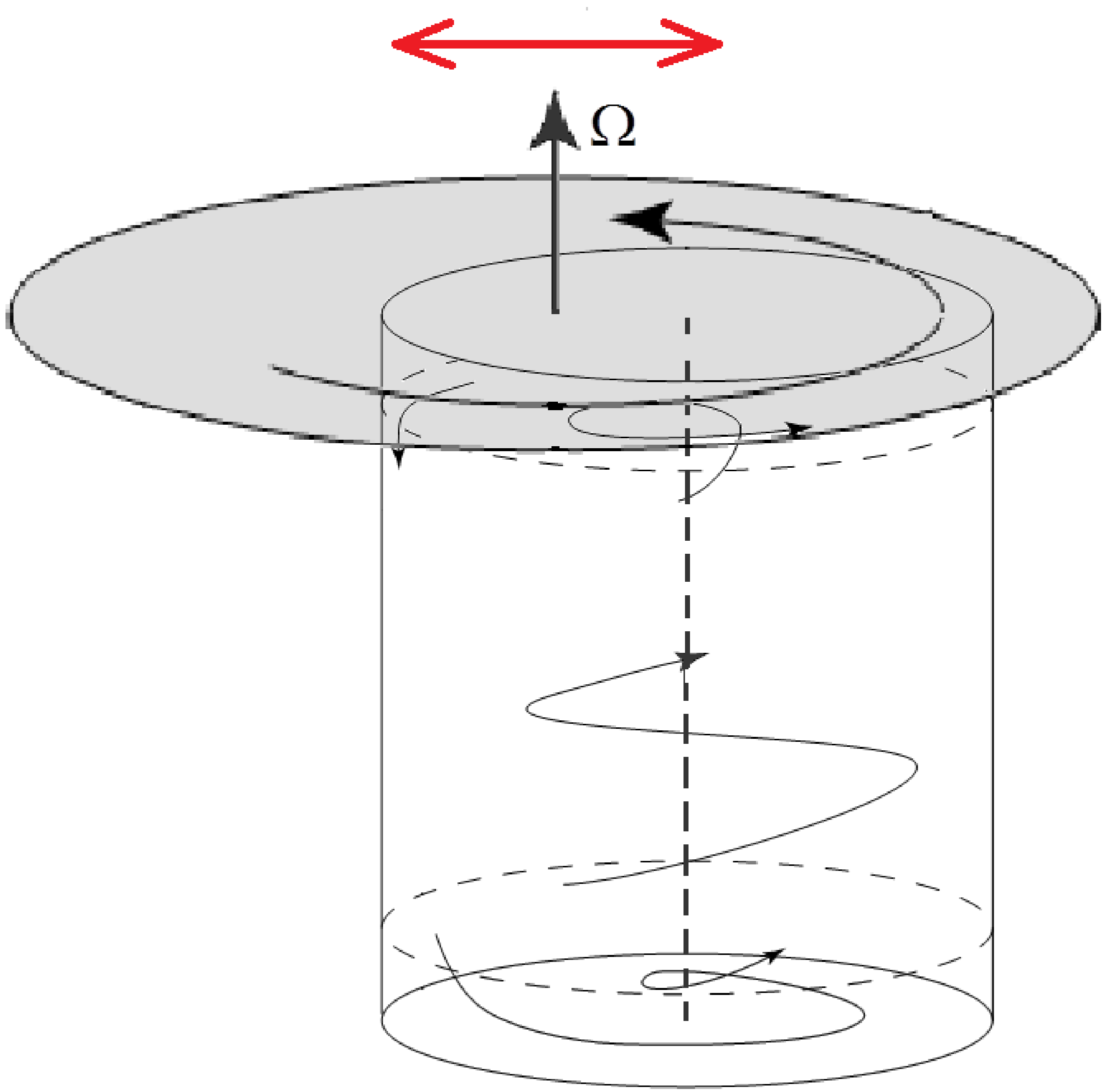




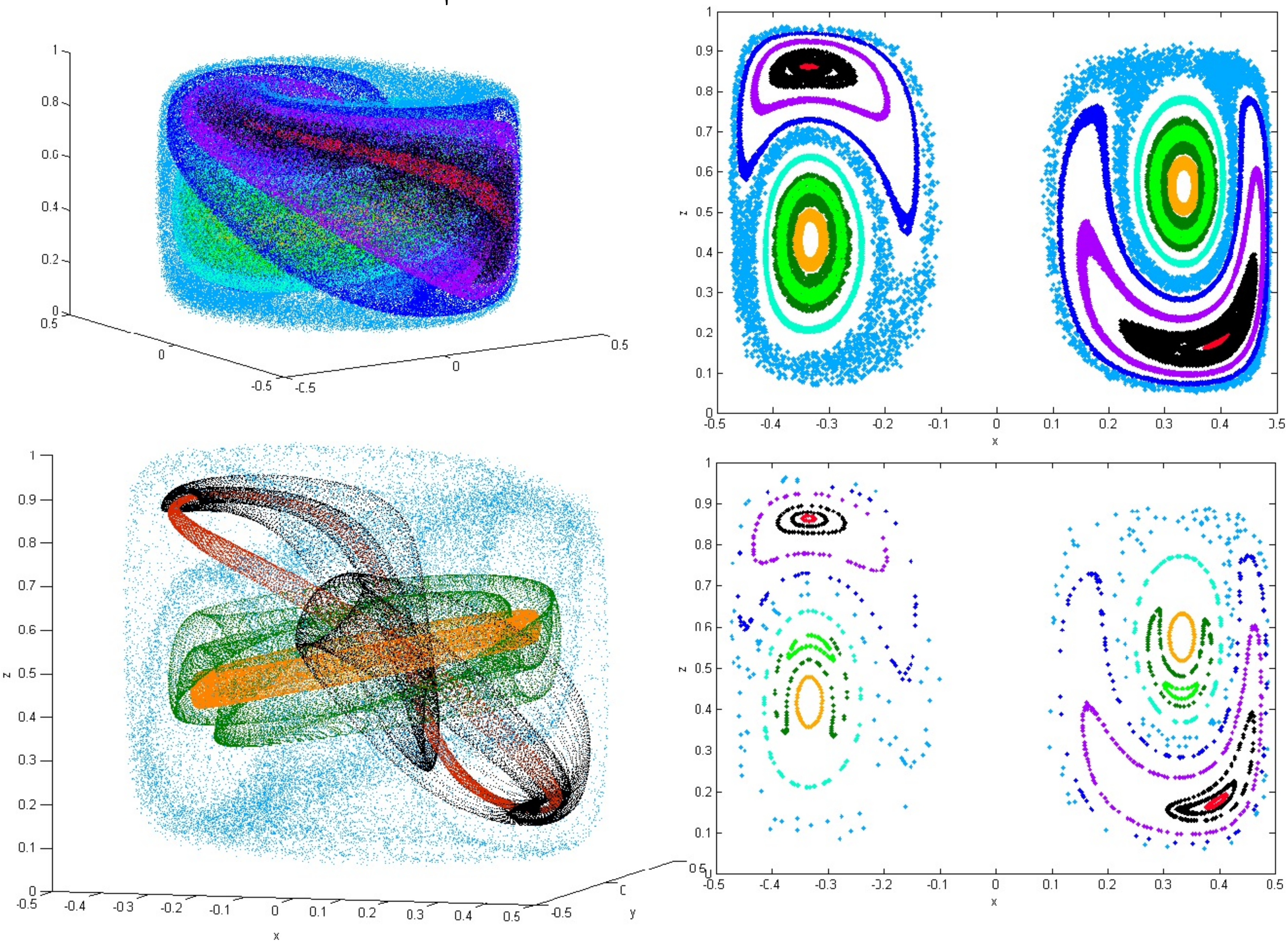
deep
eddies


0.8
0.6
y
0.4
0.2
−1.0
−0.5
0
0.5
x

Unsteady perturbation



Time-periodic disturbance



$$\begin{cases} \dot{I} = \varepsilon F^{0}(I,\phi,\theta,\sigma t) \\ \dot{\phi} = \Omega_{\phi}(I) + \varepsilon F^{1}(I,\phi,\theta,\sigma t) \\ \dot{\theta} = \Omega_{\theta}(I) + \varepsilon F^{2}(I,\phi,\theta,\sigma t) \end{cases}$$

Fourier - expand $F^{0}(I,\phi,\theta,\sigma t)$

$$\dot{I} = \varepsilon \Sigma F_{nml}(I)\sin(n\phi + m\theta + l\sigma t + \alpha_{nml})$$

$$\eta_{nml}(t) = n\phi(t) + m\theta(t) + l\sigma t + \alpha_{nml} =$$
$$n\phi_0 + m\theta_0 + \alpha_{nml} + t\left(n\Omega_{\phi}(I_0) + m\Omega_{\theta}(I_0) + l\sigma\right) + O(\delta I)$$

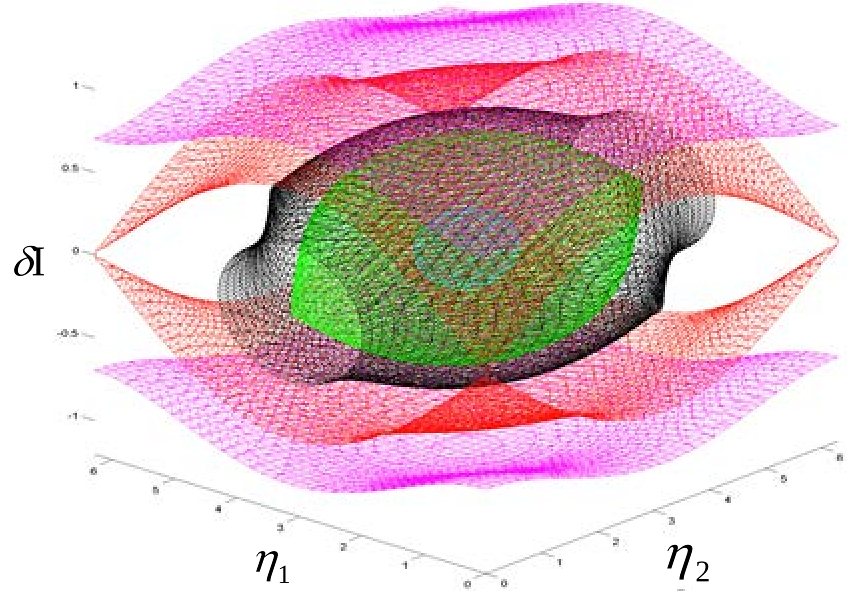
Main contribution is from $n\Omega_{\phi}(I_0) + m\Omega_{\theta}(I_0) + l\sigma = 0.$

- No resonance
- Single resonance
- Double resonance

Up to 2 independent resonant triplets {n,m,l}

Linearized system

$$
\begin{cases}
\delta\dot{I} = \varepsilon \Sigma F_{nml} \sin(\eta_{nml}) \\
\dot{\eta}_{nml} = \dfrac{(\delta I)^j}{j!} \dfrac{\partial^j (n\Omega_\phi + m\Omega_\theta)}{\partial I^j}
\end{cases}
$$

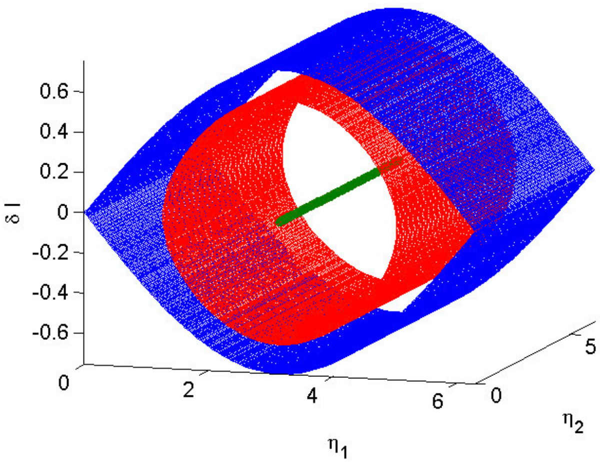


$$
G = \frac{\delta I^{j+1}}{(j+1)!} + \varepsilon \Sigma \frac{F_{nml}(I_0)\cos(\eta_{nml})}{\partial^j (n\Omega_\phi + m\Omega_\theta)/\partial I^j}
$$

We can estimate resonance width $\Delta I = 2\left(\Sigma \dfrac{\varepsilon (j+1)!\, F_{nml}(I_0)}{\dfrac{\partial^j (n\Omega_\phi + m\Omega_\theta)}{\partial I^j}} \right)^{\frac{1}{j+1}}$

and we can map out the flow geometry near resonances

$$G=\frac{\delta I^{\,j+1}}{(j+1)!}+\sum\frac{\varepsilon F_{nml}(I_0)\cos(\eta_{nml})}{\partial^j\left(n\Omega_\phi+m\Omega_\theta\right)/\partial I^j}$$

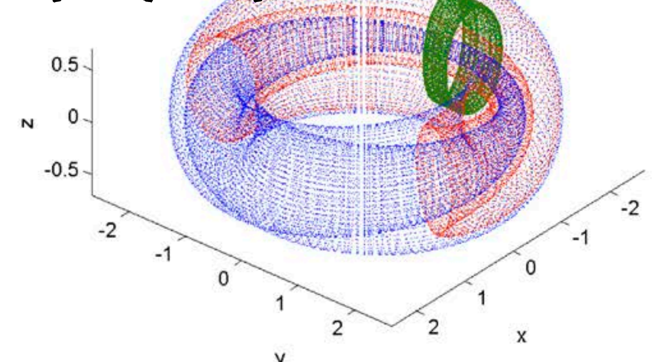


Need to convert from $\{\delta I,\eta\}$ to {x,y,z}.

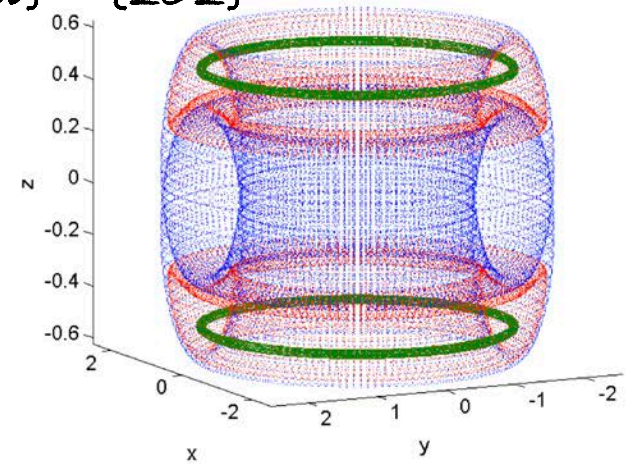
Assume nested horizontal tori with circular cross-section:

$$\begin{cases} x=(R+r\cos\phi)\cos\theta \\ y=(R+r\cos\phi)\sin\theta,\; r=r_0+\delta r=\sqrt{r_0^{\,2}+\delta I/\pi} \\ z=r\sin\phi \end{cases}$$

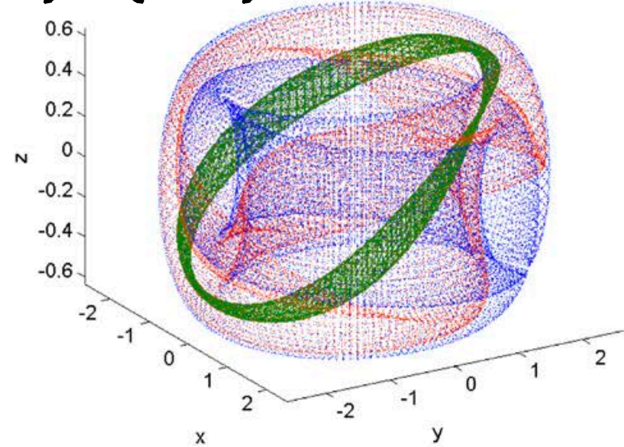
{nml}={011}



{nml}={101}

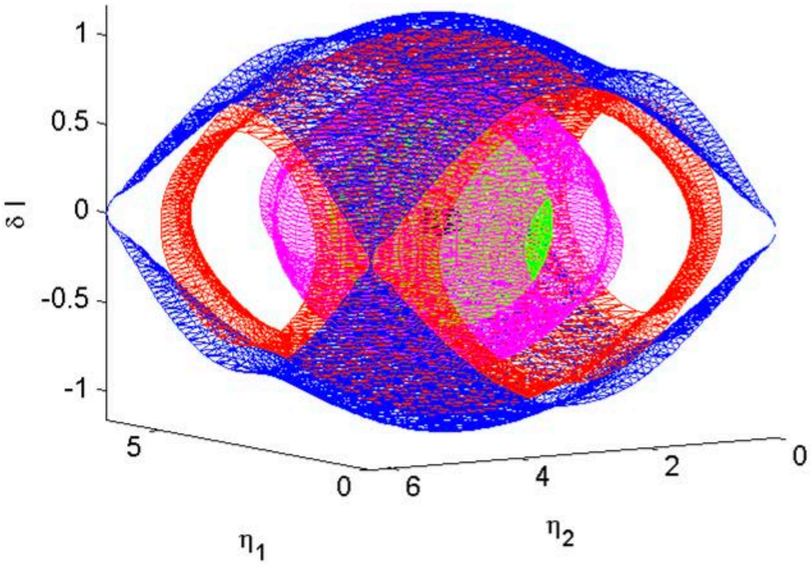


{nml}={111}

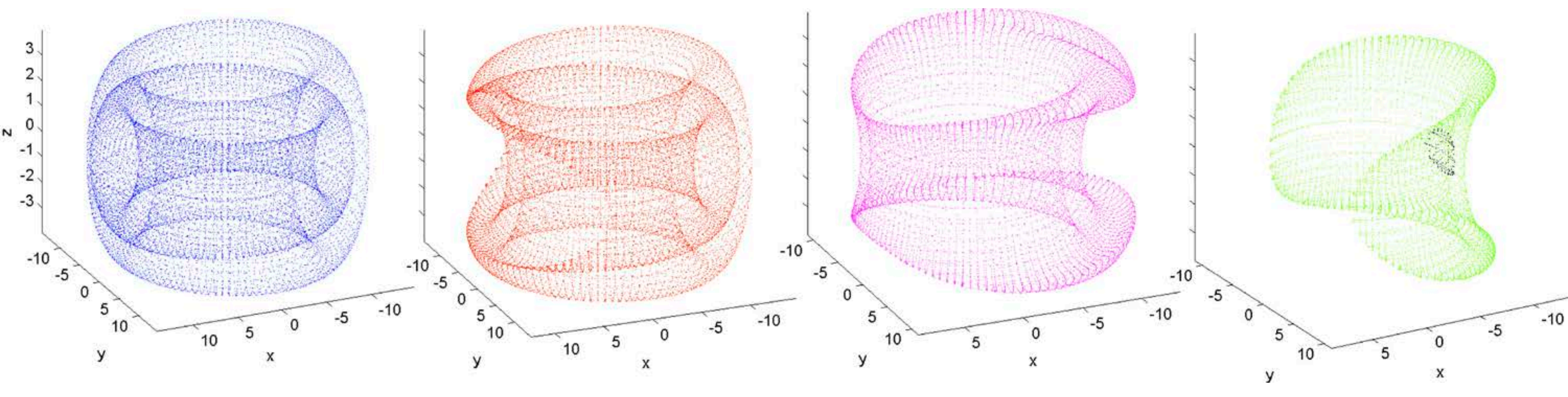


Double resonance geometry

$\{n_1, m_1, l_1\} = \{1, 0, 1\}$
$\{n_2, m_2, l_2\} = \{0, 1, 1\}$

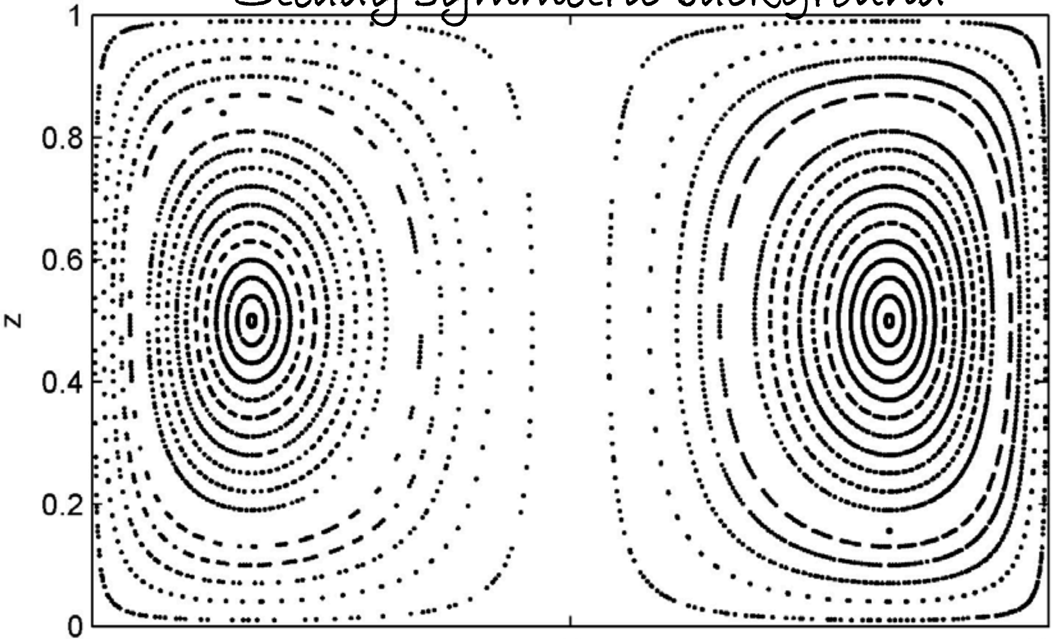


- tori
- pretzels
- spheres

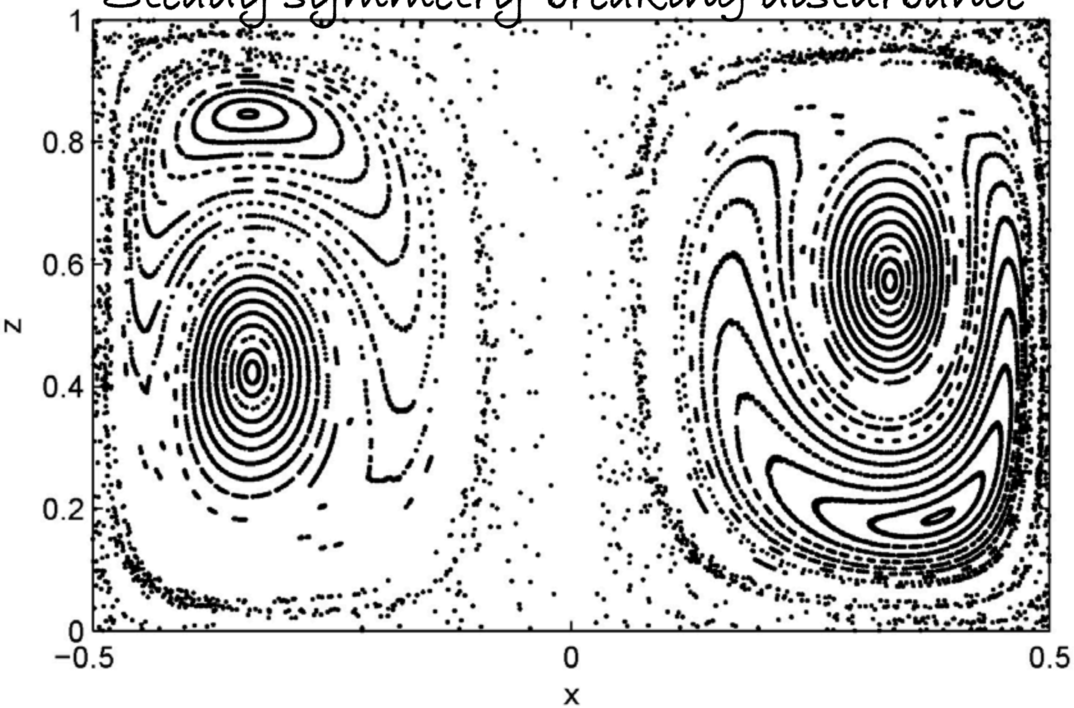


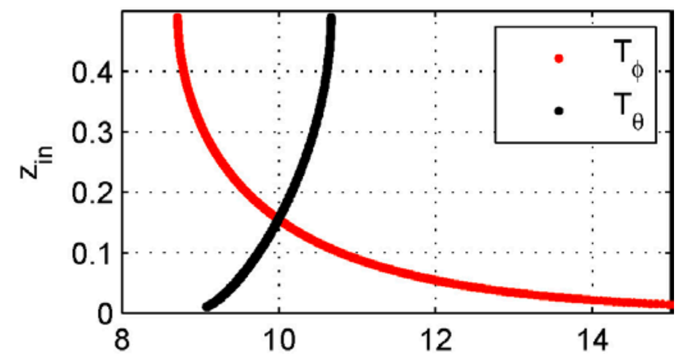
Phenomenological model

Steady symmetric background



Steady symmetry-breaking disturbance

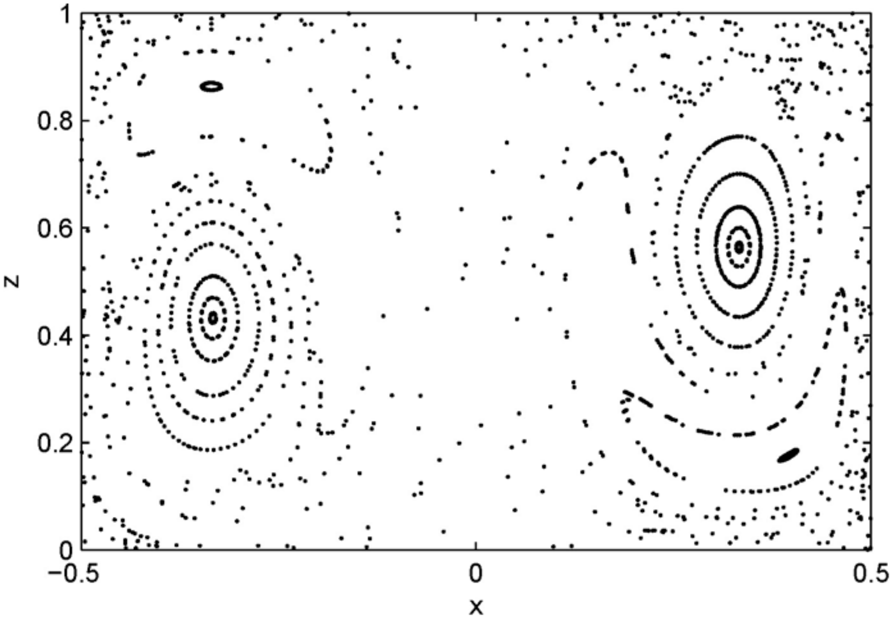


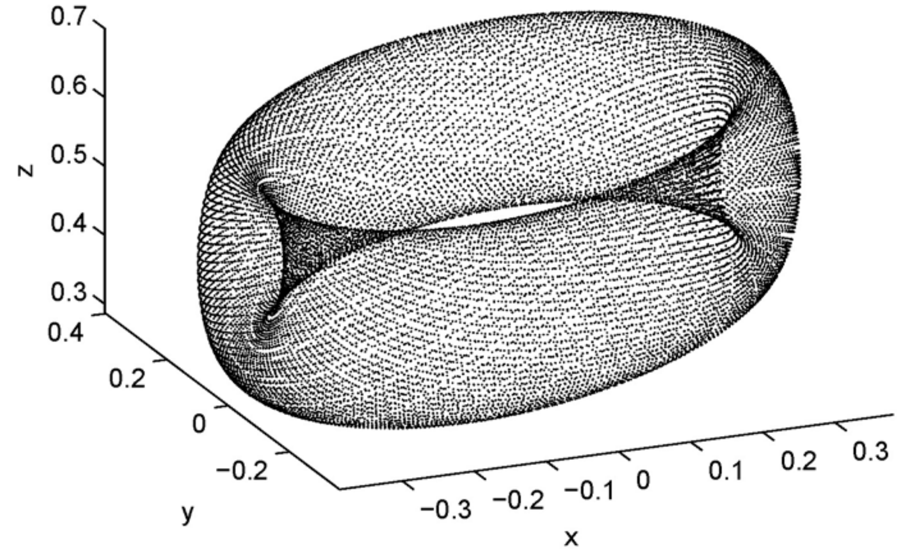


$$n\Omega_\phi + m\Omega_\theta + l\sigma = 0$$

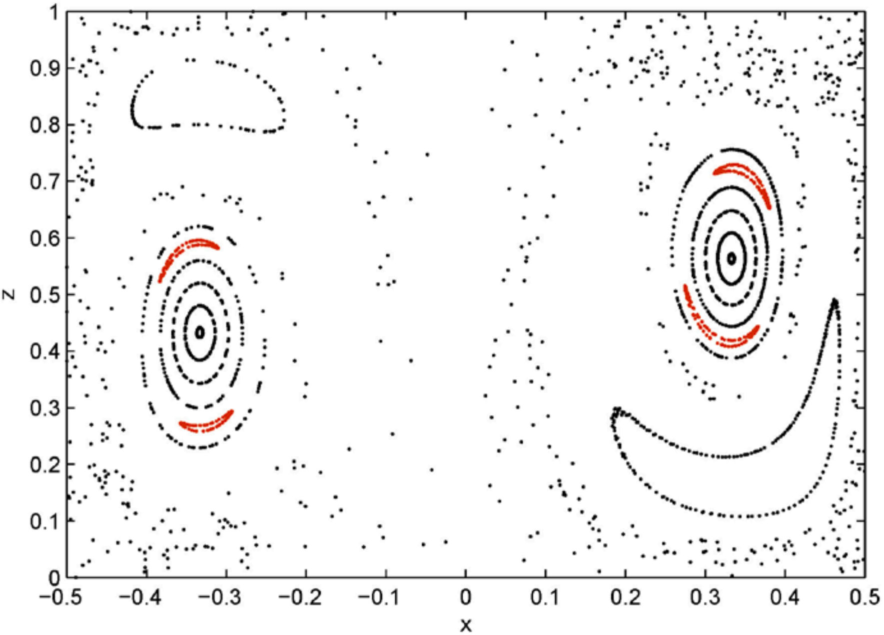
1) T=1: non-resonant
2) T=4.5: {2,0,1}
3) T=11: {0,1,1}

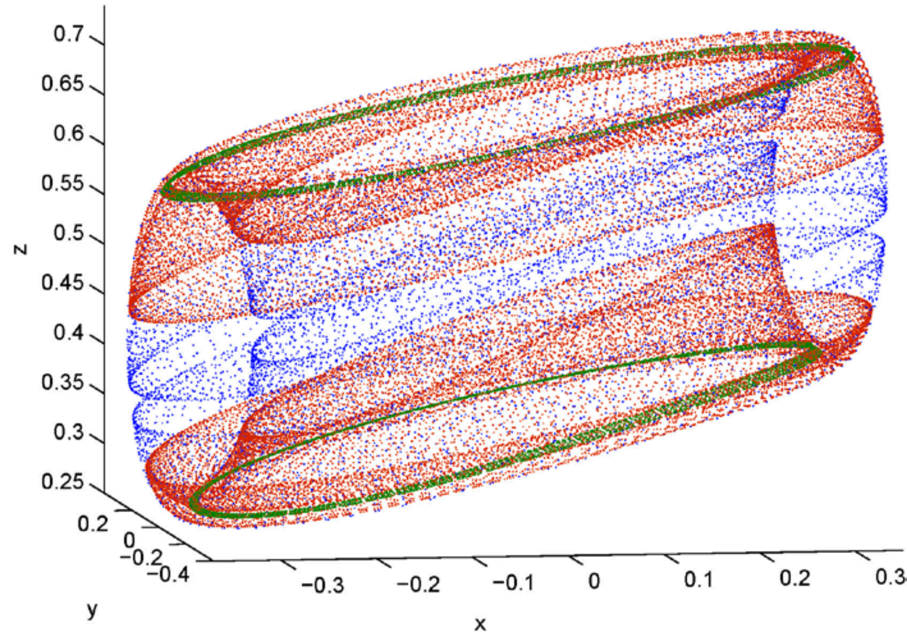
1) T=1: non-resonant

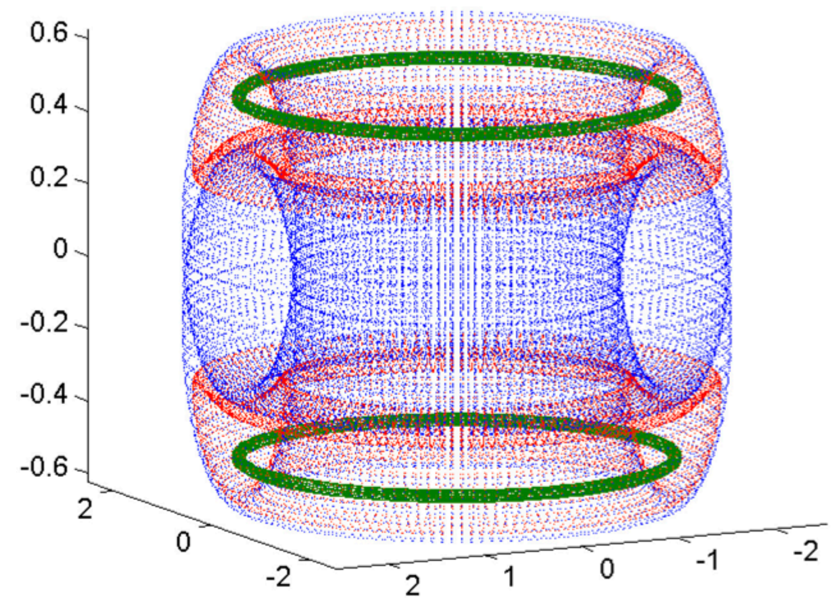




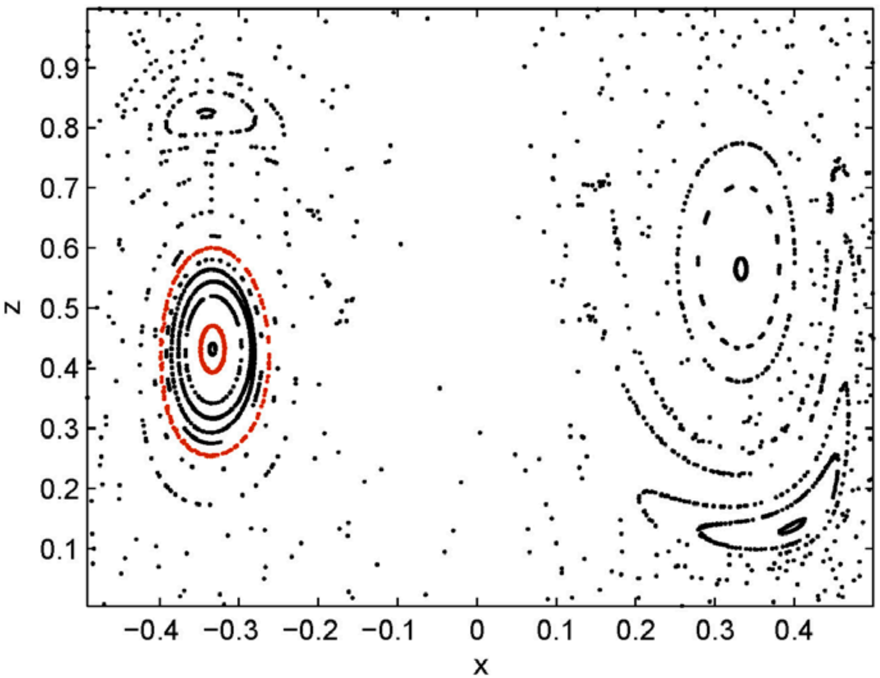
1) T=4.5: {n,m,l}={2,0,1}



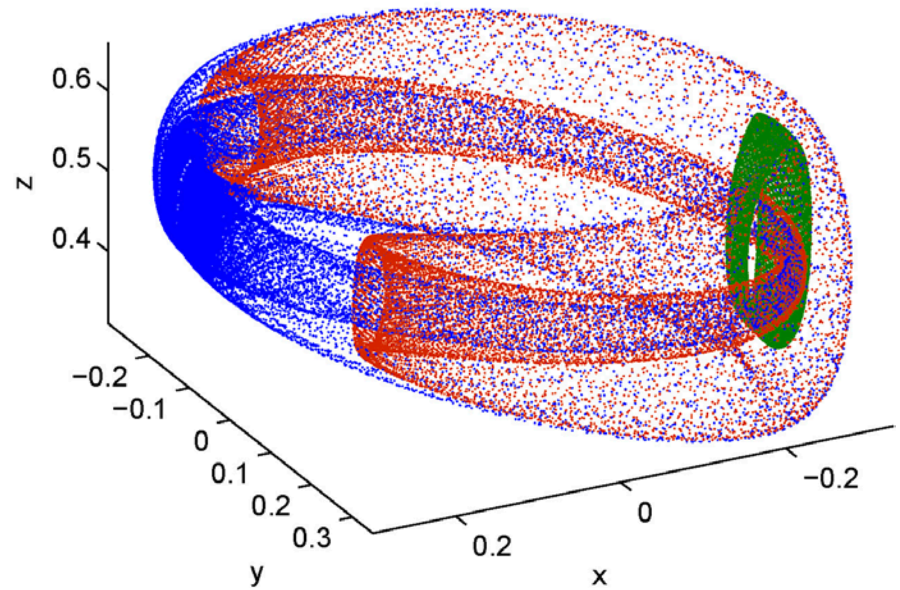




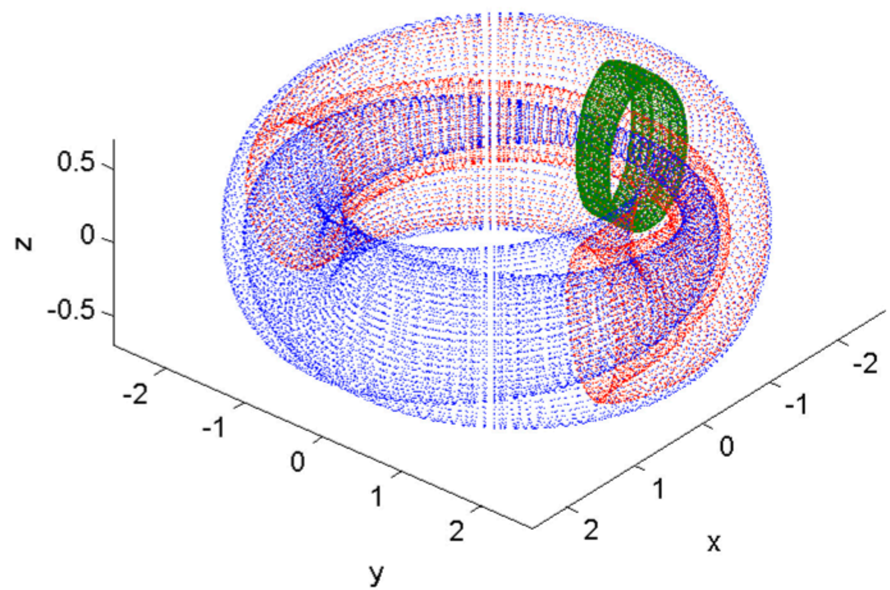
1) T=11: {n,m,l}={1,0,1}



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
z
−0.4 −0.3 −0.2 −0.1 0 0.1 0.2 0.3 0.4
x



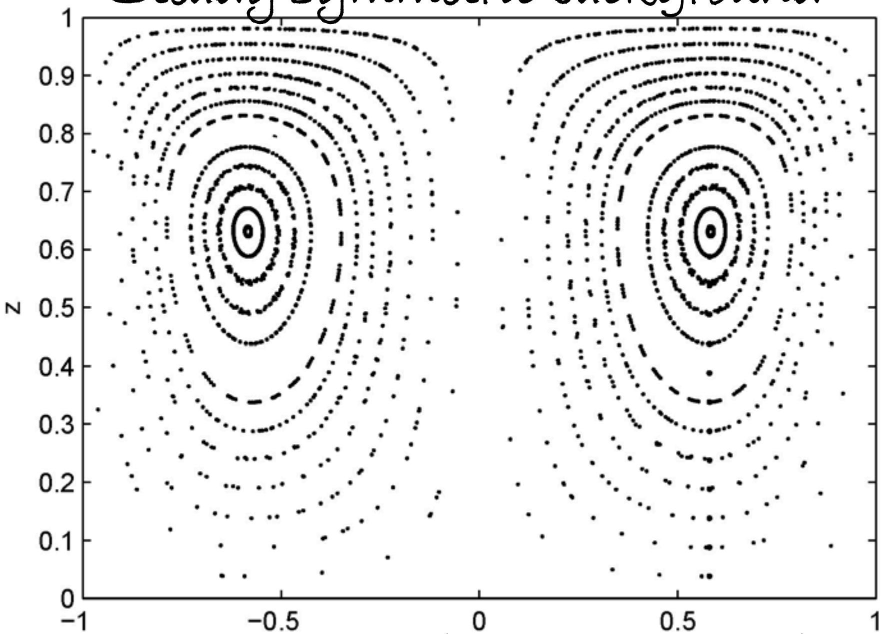
0.6
0.5
0.4
z
−0.2
−0.1
0
0.1
0.2
0.3
y
0.2
0
−0.2
x



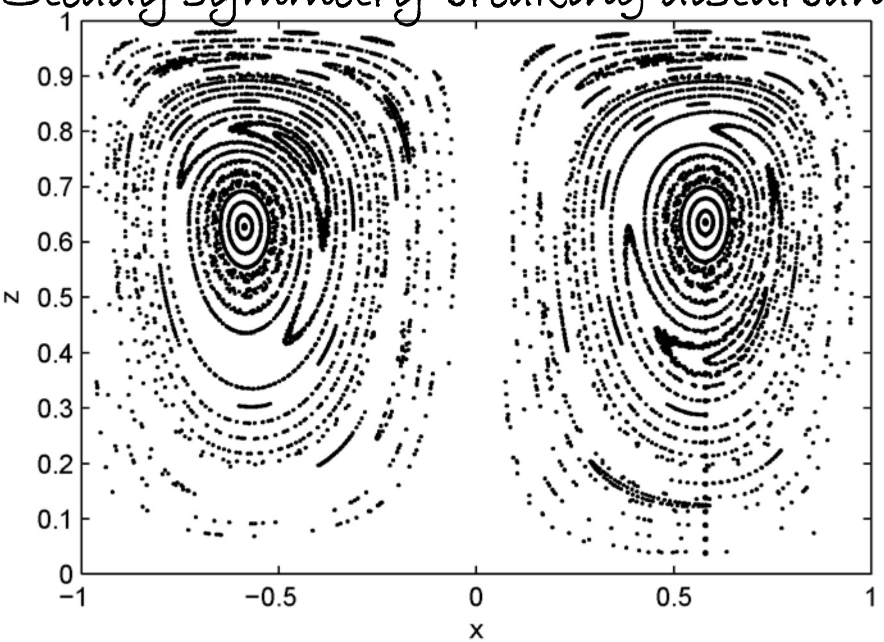
0.5
0
−0.5
z
−2
−1
0
1
2
y
2
1
0
−1
−2
x

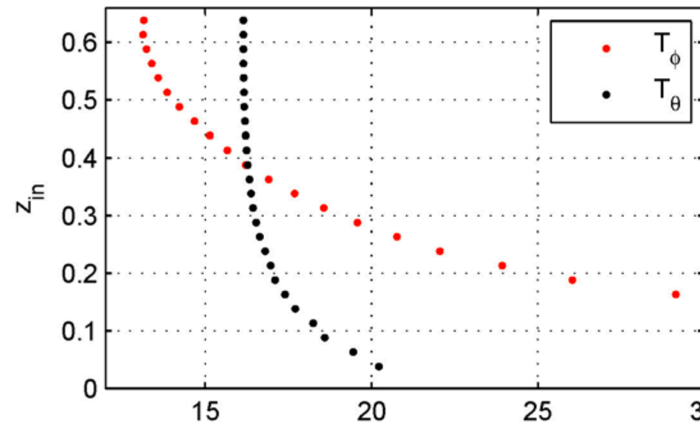
Full numerical solution

Steady symmetric background



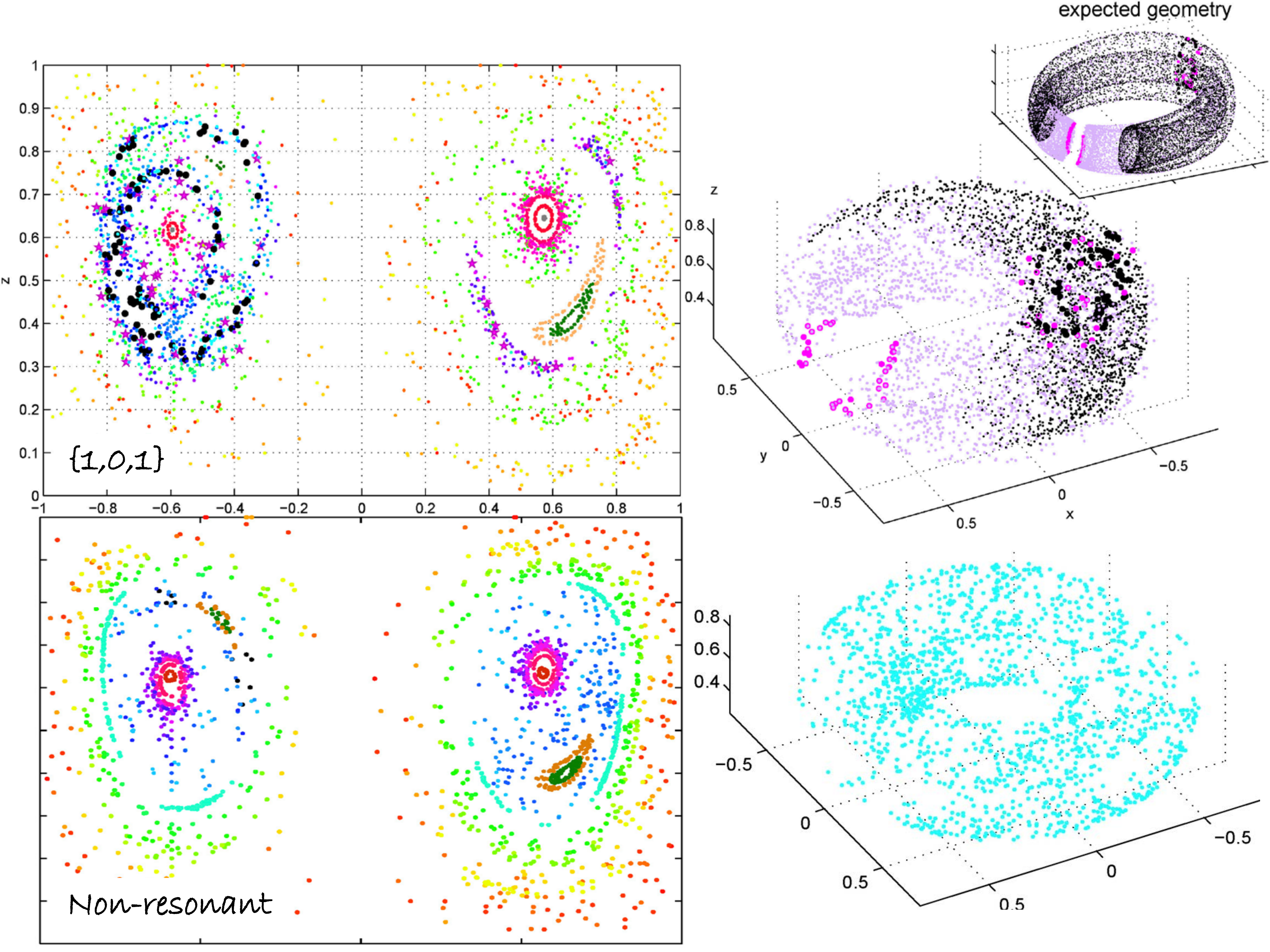
Steady symmetry-breaking disturbance





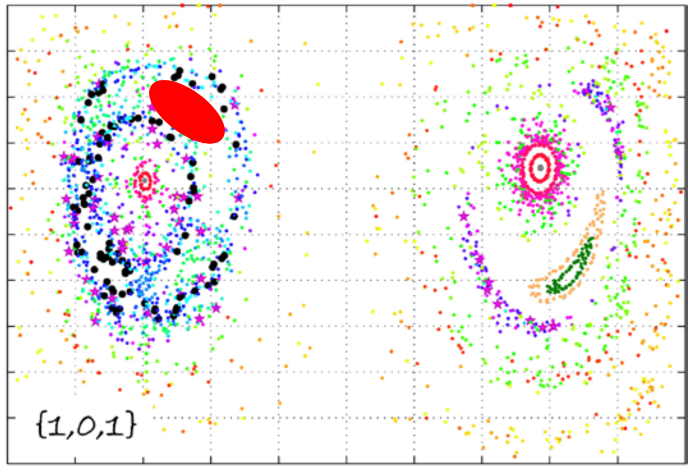
$$n\Omega_\phi + m\Omega_\theta + l\sigma = 0$$

1) T=14.5 non-resonant
2) T=16.35: {1,0,1}

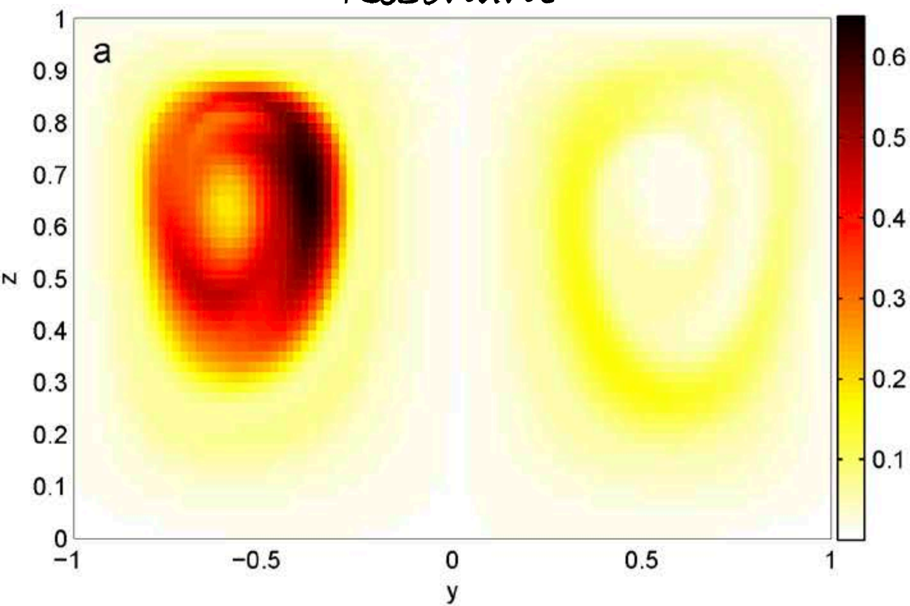


expected geometry
{1,0,1}
Non-resonant
z
x
y

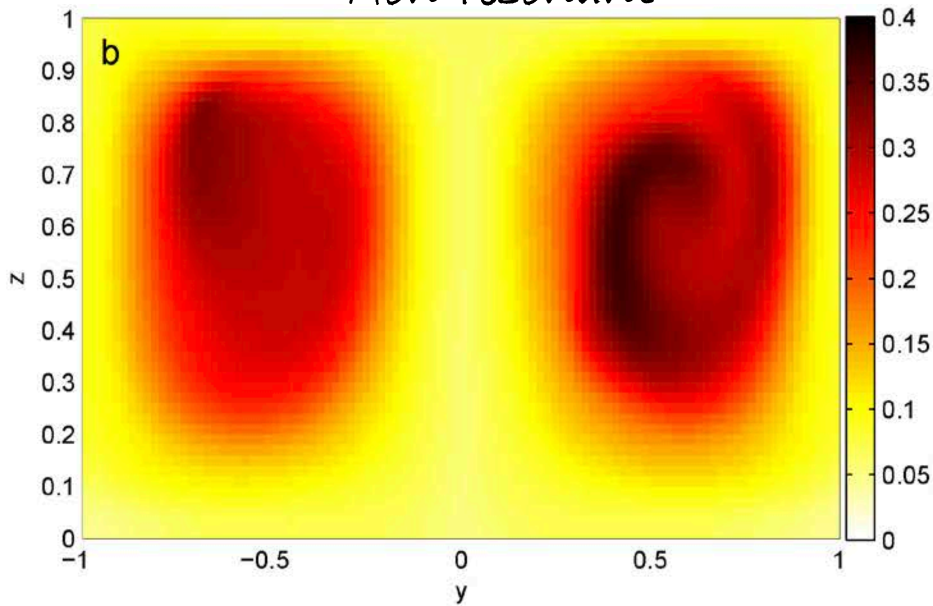
Dye release experiment



Resonant



Non-resonant



Summary

- Studied chaotic advection in steady and nonsteady 3d rotating can flow
- Resonances affect barriers and flow geometries
- Developed theoretical framework for describing flows near resonances
- Mapped out possible Lagrangian geometries
- Tested theoretical predictions using both phenomenological model and full numerical solution
- Resonances have strong influence on dye and other tracers