Model error in the rotating can

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Rotating can as testbed

Two models:

- kinematic
  3D velocity non-divergent but no dynamics
- CFD
  nonlinear numerical model, Navier-Stokes equations

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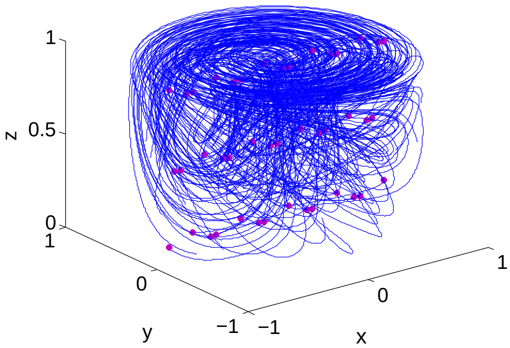
However:

- similar coherent structures (Poincaré sections and/or FTLEs) can be found in both systems

Trajectories from Tamay



- u, v, w from Navier Stokes with “wind” forcing
- Reynolds number, Re=100

Kinematic representation of velocity field

(Pratt and Rypina)

$$
\begin{aligned}
u(x,y,z) &= -\frac{x}{3}(1-2z)(1-r) - \alpha y - \frac{1}{2}\epsilon(1-\beta z)(1-r^2-2y^2) \\
v(x,y,z) &= -\frac{y}{3}(1-2z)(1-r) + \alpha x - \epsilon(1-\beta z)xy \\
w(x,y,z) &= z(1-z)(\frac{2}{3}-r)
\end{aligned}
$$

- $r^2 = x^2 + y^2$
- non-divergent, satisfy no-normal flow bc at $r = 1$, $z = 0, 1$
- ϵ controls strength of horizontal velocity at origin
- β controls strength of vertical structure
- α controls strength of horizontal swirling

Model bias, Kennedy-O'Hagan framework

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$$\mathbf{u}_{CFD}(x, y, z) = \mathbf{u}_{KM}(x, y, z, \alpha) + \mathbf{u}_{B}(x, y, z, \alpha)$$

OR

$$\mathbf{u}_{B}(x, y, z, \alpha) = \mathbf{u}_{CFD}(x, y, z, \alpha) - \mathbf{u}_{KM}(x, y, z)$$

GASP framework

- input vector (*i* design points, *d* dimensions)

$$x_i = \{x_{i1}, ..., x_{id}\}$$

- simulation response at design points

$$y_i(\mathbf{x}_i) = \beta + z(\mathbf{x}_i)$$

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$$\text{mean } \beta \qquad E[z(\mathbf{x}_i)] = 0 \qquad Var[z(\mathbf{x}_i)] = \sigma_z^2$$

- correlation structure

$$R_{ij} = corr[z(\mathbf{x}_i), z(\mathbf{x}_j)] = \prod_{k=1}^{d} \exp\{-\theta_k (x_{ik} - x_{jk})^{1.9}\}$$

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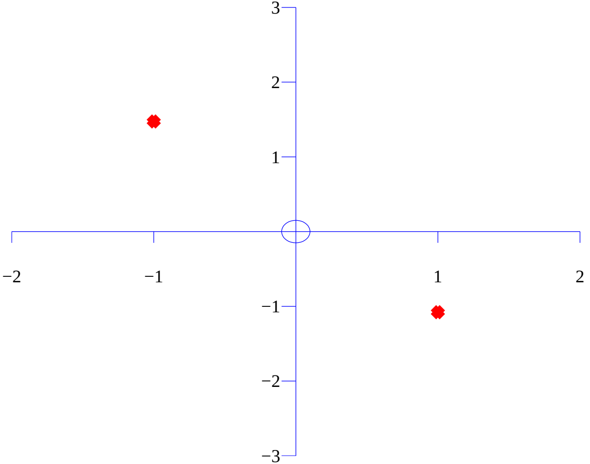
- BLUP (mean and standard error)

$$\hat{y}(\tilde{\mathbf{x}}) = \beta + \mathbf{r}'\mathbf{R}^{-1}(y(\tilde{\mathbf{x}}) - \beta)$$

$$s^2(\tilde{\mathbf{x}}) = \sigma_z^2 (1 - \mathbf{r}'\mathbf{R}^{-1}\mathbf{r}' + \frac{(1 - \mathbf{1}'\mathbf{R}^{-1}\mathbf{r}')^2}{\mathbf{1}'\mathbf{R}^{-1}\mathbf{1}'})$$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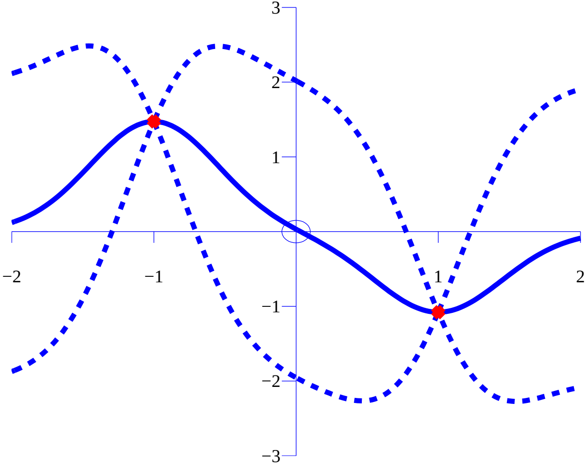
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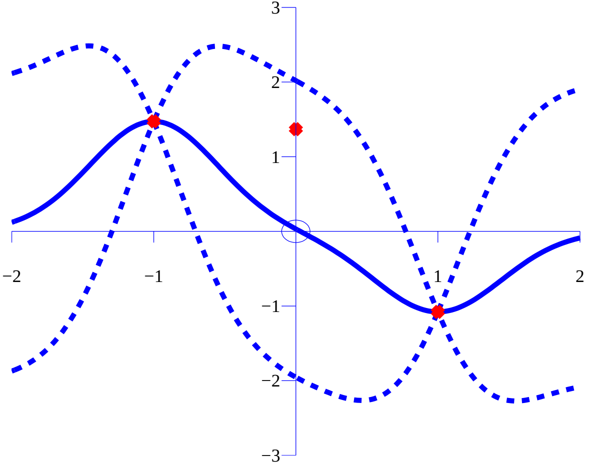
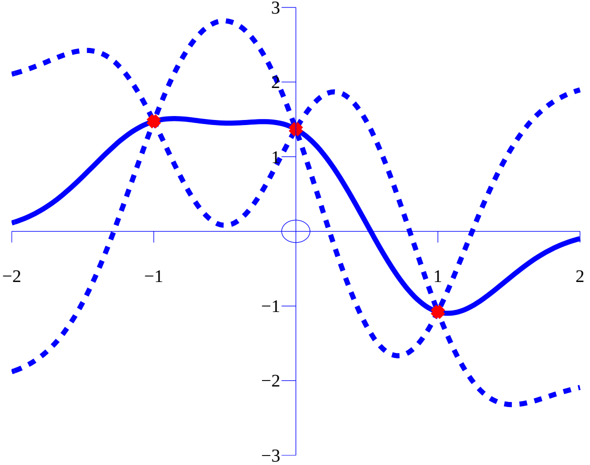
GASP cartoon *N. Lawrence's talk, "Learning & Inference w/GaSPs" (2005)*





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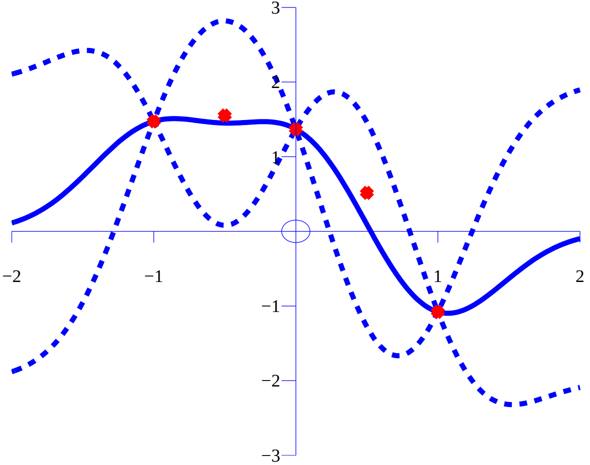

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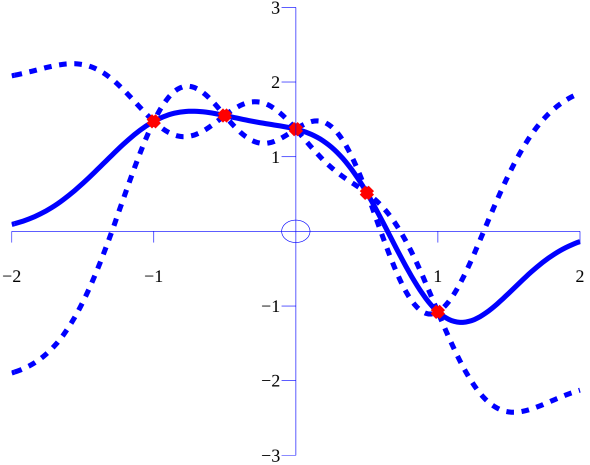

GASP framework, translation

- input vector – (x, y, z, α) evaluate $u_{CFD} - u_{KM}$ at N locations
$\{(x_1, y_1, z_1, \alpha_1), (x_2, y_2, z_2, \alpha_2), \ldots, (x_N, y_N, z_N, \alpha_N)\}$

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- find $\boldsymbol{\theta}$'s that maximize $\mathcal{L}(\boldsymbol{\theta}) = p(\mathbf{u}_B | \boldsymbol{\theta}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\alpha})$

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- untested scenario – $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\alpha})$ not used as input
- how “close” is that scenario to those evaluated above

$$\hat{\mathbf{r}} = (R((\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\alpha}), (x_1, y_1, z_1, \alpha_1)), \ldots, R((\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\alpha}), (x_N, y_N, z_N, \alpha_N)))^T$$

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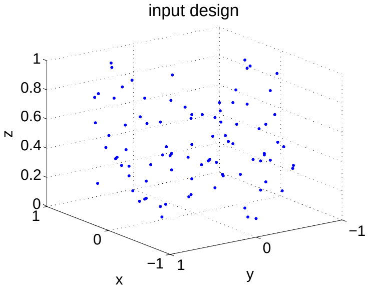
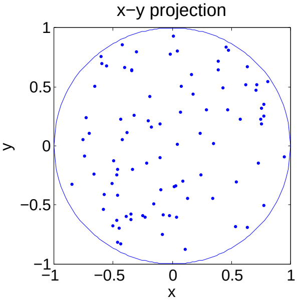
- GaSP function – expected value of bias at the $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\alpha})$

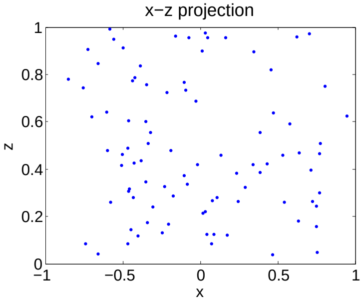
$$\hat{u}_B(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\alpha}) = \hat{\beta} + \hat{\mathbf{r}}^T \hat{\mathbf{R}}^{-1} (\mathbf{u}_B - \hat{\beta})$$

Strategy: design of inputs



- data: sample CFD velocities at "spread out" locations around can (inputs)



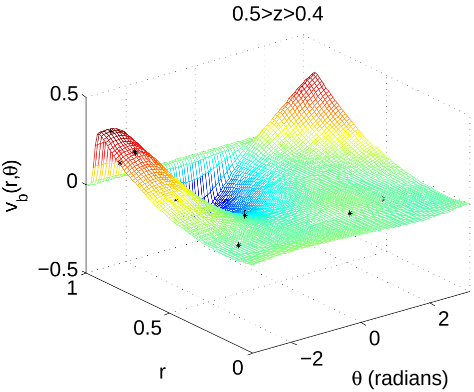


Strategy

- data: sample CFD velocities at “spread out” locations around can (inputs)
- evaluate kinematic model at input locations and calculate bias (responses)
- fit GaSP as model of bias for each u, v, w
- hard to fit, more natural w/inputs in cylindrical coordinates

  tricky, need periodic correlation (Spiller et al, submitted to SIAM JUQ)

velocity bias at various depths



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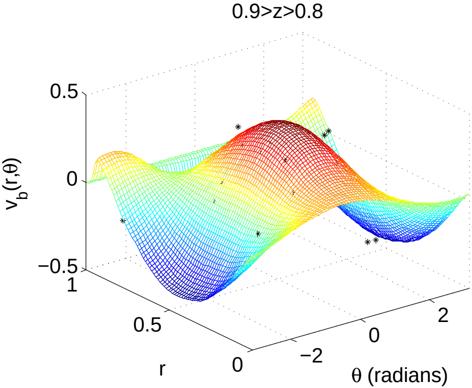

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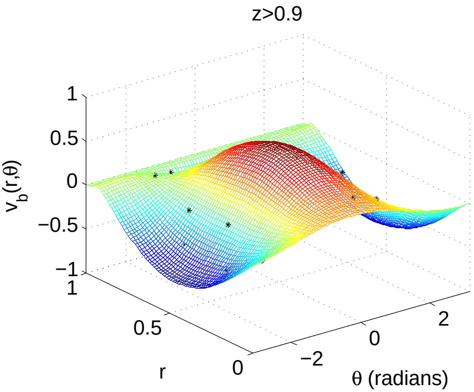
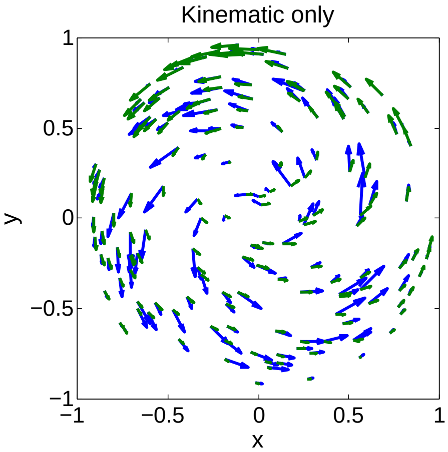


Comparison of velocity fields



- CFD field and“best fit” Kinematic field
- (u, v) projection of top of can, $z > 0.8$

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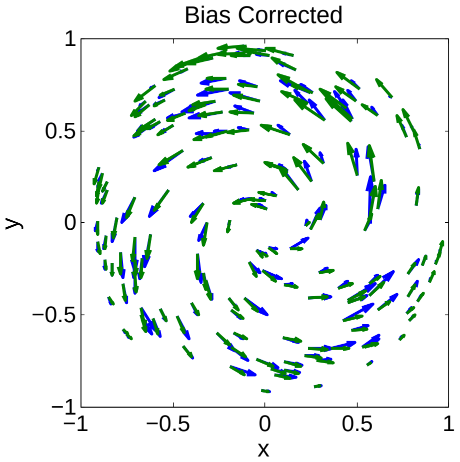

- CFD field and bias corrected kinematic field
- (u, v) projection of top of can, $z > 0.8$

Comparisons of trajectories

- Kinematic model

$$\frac{dx}{dt} = u_{KM}, \quad \frac{dy}{dt} = v_{KM}, \quad \frac{dz}{dt} = w_{KM}$$

- CFD model

$$\frac{dx}{dt} = u_{CFD}, \quad \frac{dy}{dt} = v_{CFD}, \quad \frac{dz}{dt} = w_{CFD}$$

- Bias corrected kinematic model (labeled RF)

$$\frac{dx}{dt} = u_{KM} + \hat{u}_B, \quad \frac{dy}{dt} = v_{KM} + \hat{v}_B, \quad \frac{dz}{dt} = w_{KM} + \hat{w}_B$$

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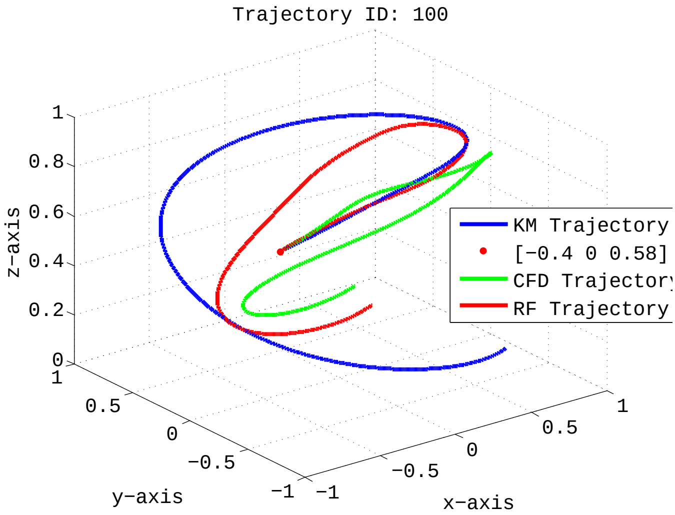

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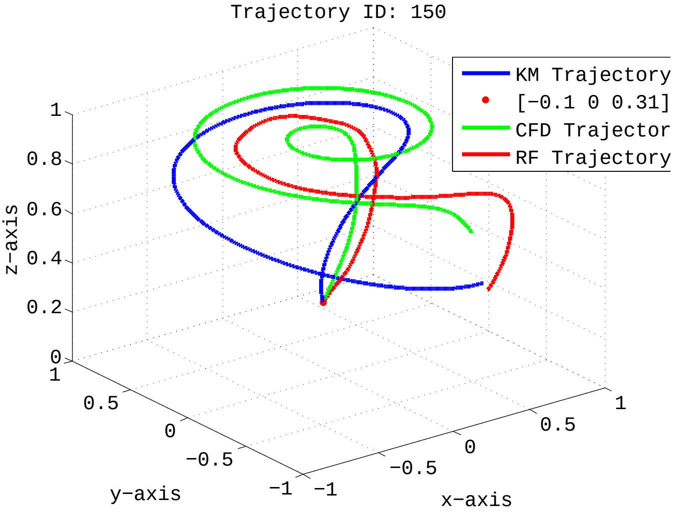

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