Toy Model
Vortex Chain + Oscillations+Ekman Pumping

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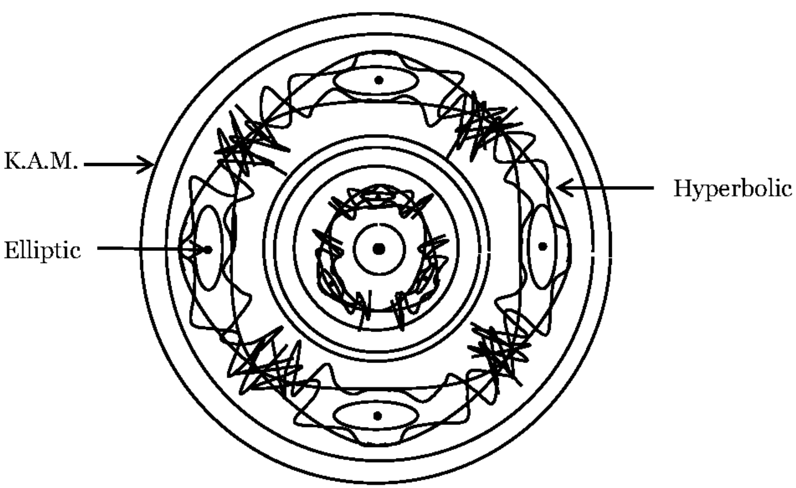
Perturbed Hamiltonian Systems

- 2DOF
- 3DOF

2D steady + time
2D steady + 3rd

KAM Tori=Barriers
Chaos & Regular zones
Action-Angle

KAM Tori $\neq$ Barriers
Arnold Diffusion
2Actions-2Angles



Toy Models

- 2DOF
- 3DOF

Can Model

$$\dot{I} = \varepsilon F_1(I, \phi_1, \phi_2, t)$$
$$\dot{\phi}_1 = \Omega_1(I) + \varepsilon F_2(I, \phi_1, \phi_2, t)$$
$$\dot{\phi}_2 = \Omega_1(I) + \varepsilon F_3(I, \phi_1, \phi_2, t)$$

Action-Angle-Angle

1 Slow & 2 Fast Variables

Vortex Model

$$\dot{I}_1 = \varepsilon F_1(I, \phi_1, \phi_2, t)$$
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Action-Action-Angle

2 Slow & 1 Fast Variables

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Action-Action-Angle

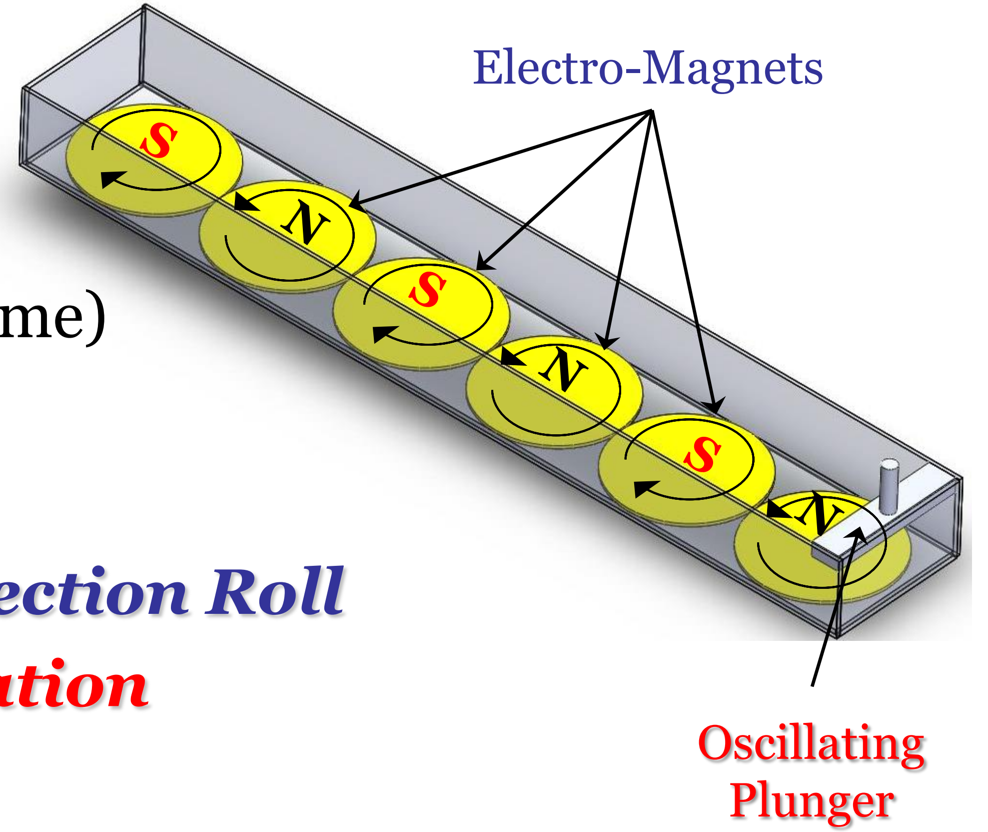
1 Slow & 2 Fast Variables

Physical Model & Assumptions

- Newtonian Incompressible Fluid
- Slightly 3-D Open Flow
(Thin Electrolyte Layer)
- No Waves are Excited
(Excitation period $\ll$ Viscous time)
- M.H.D. Technique

- ***Rayleigh-Bénard Convection Roll***
- ***Time Dependent Oscillation***
- ***3-D Ekman Pumping***

- System Diagram



[1] T.H. Solomon and I. Mezic, Nature **425**, 376-380 (2003)

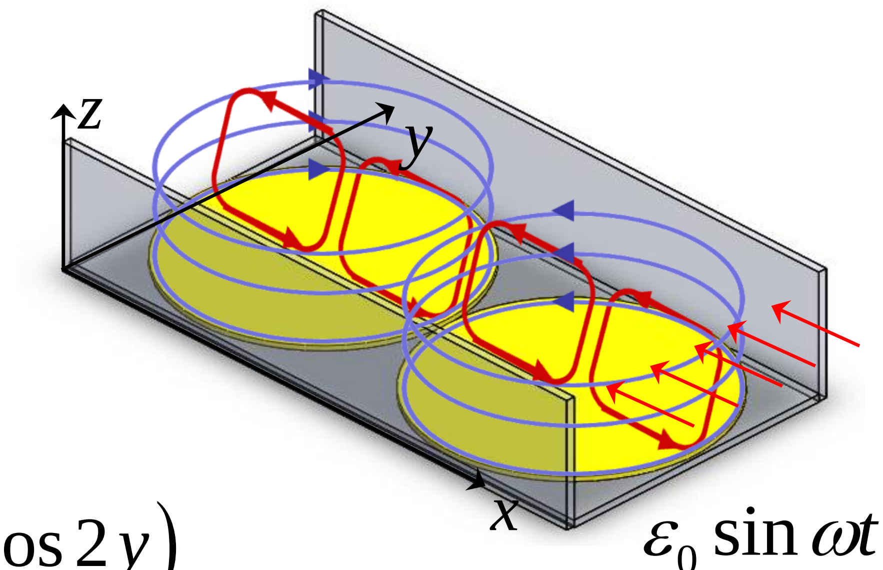
Velocity Field/Dynamical System

- Dynamics of an Advected Particle

$$\dot{x} = \cos\left(x + \varepsilon_0 \sin \omega t\right)\sin y$$
$$+\varepsilon_1 \sin\left(2x + 2\varepsilon_0 \sin \omega t\right)\sin z$$
$$\dot{y} = -\sin\left(x + \varepsilon_0 \sin \omega t\right)\cos y$$
$$+\varepsilon_1 \sin 2y \sin z$$
$$\dot{z} = 2\varepsilon_1 \cos z\left(\cos\left(x + \varepsilon_0 \sin \omega t\right) + \cos 2y\right)$$

- Flow Structure

[1]



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Unperturbed Case

- Action-Action-Angle
- Physical Space

$$\varepsilon_0 = 0 \quad \varepsilon_1 = 0$$

$\dot{\psi} = 0$ Action

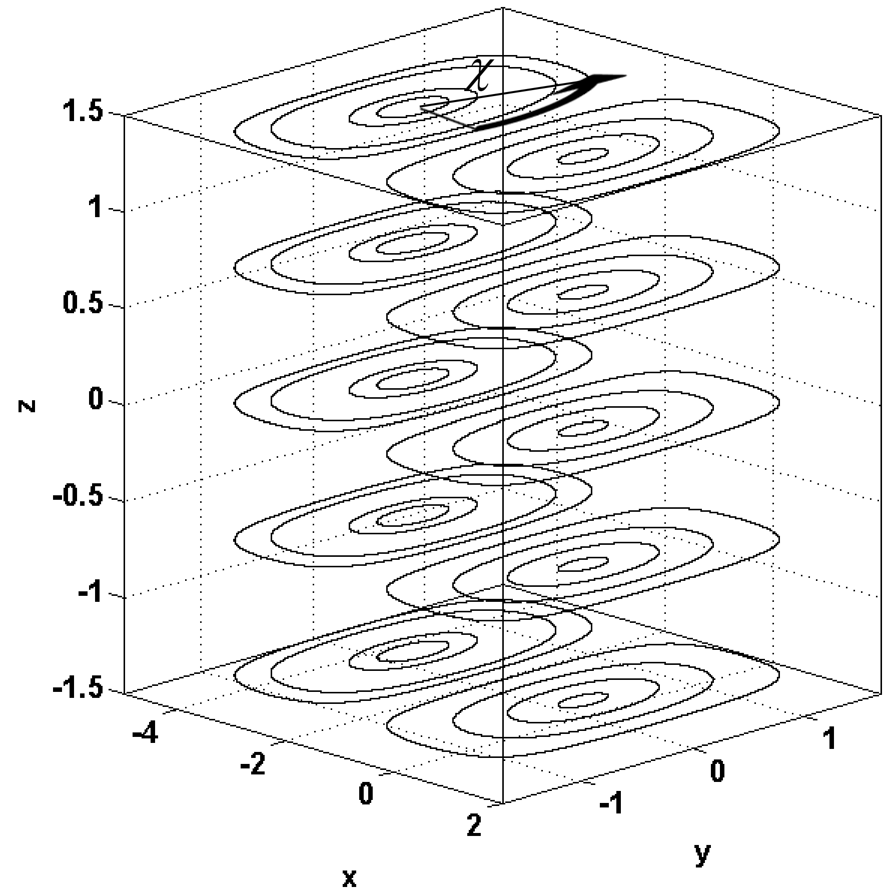
$\dot{z} = 0$ Action

$\dot{\chi} = \Omega(\psi)$ Angle

$$\psi = \cos x \cos y$$

$$T(\psi) = 4 \int_0^{x_{\max}} \left(\cos^2(x) - \psi^2\right)^{-1/2} dx$$

$$x_{\max} = \frac{1}{\cos\psi} \qquad \Omega(\psi) = 2\pi / T(\psi)$$



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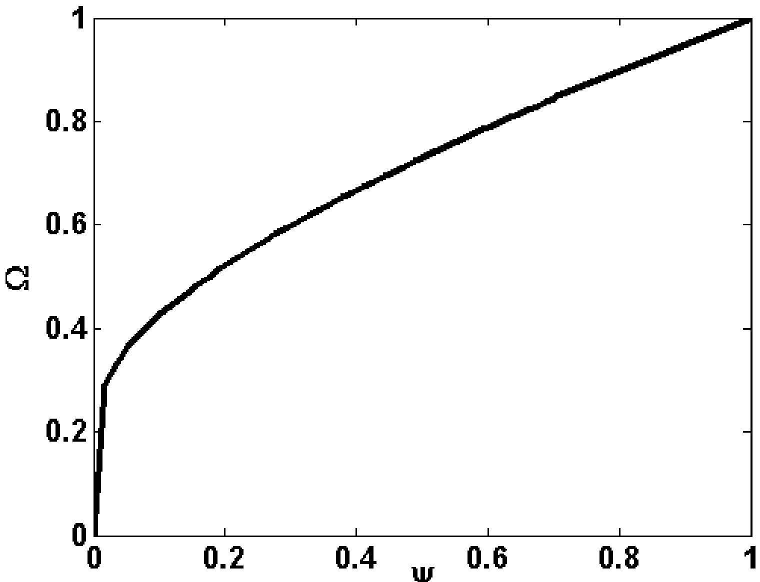

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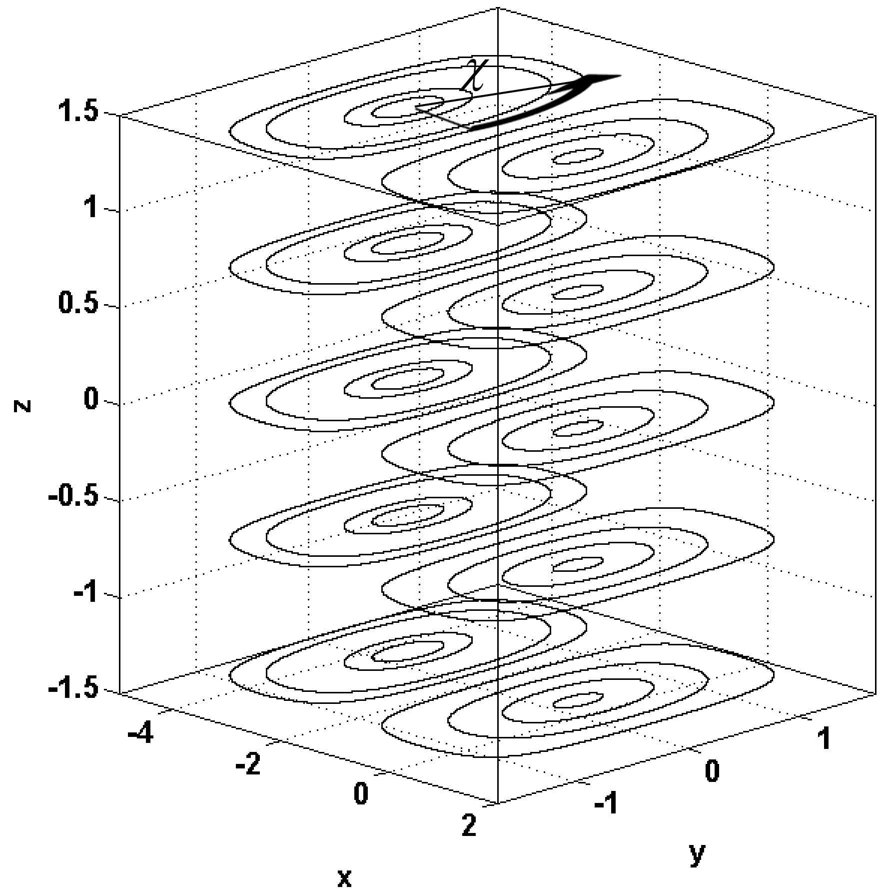


Perturbed Case

- Perturbation of Action-Action-Angle

$$\omega = O(\Omega) = O(1) \qquad \varepsilon_0 / \varepsilon_1 = O(1)$$

$$\dot{\psi} = -\frac{1}{2}\varepsilon_0 \sin(\omega t) G_0(\psi, \chi) - 2\varepsilon_1 \sin z\psi G_1(\psi, \chi)$$

$$\dot{z} = 2\varepsilon_1 \cos z H(\psi, \chi)$$

$$\dot{\chi} = \Omega(\psi) + \varepsilon_0 I_0(\psi, z, \chi) + \varepsilon_1 I_1(\psi, z, \chi)$$

$$G_0(\psi, \chi) = \sin 2y$$

$$H(\psi, \chi) = \cos 2y + \cos 2y$$

$$G_1(\psi, \chi) = \sin^2 x + \sin^2 y$$

(ψ, z) 2 Slow Variables

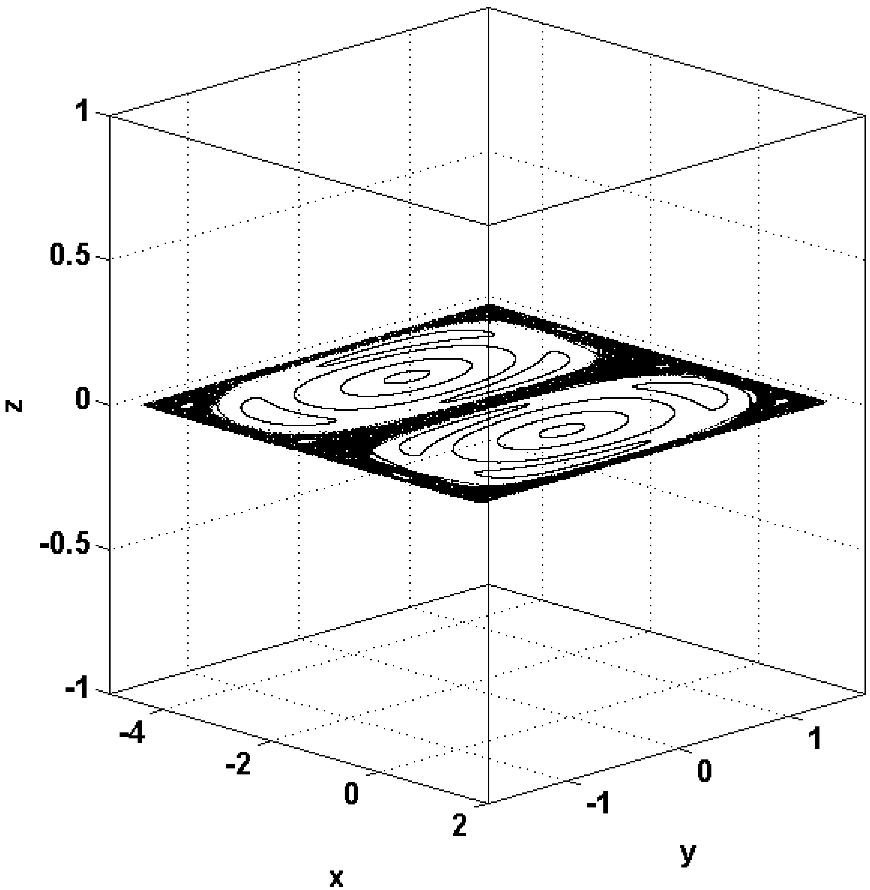
+

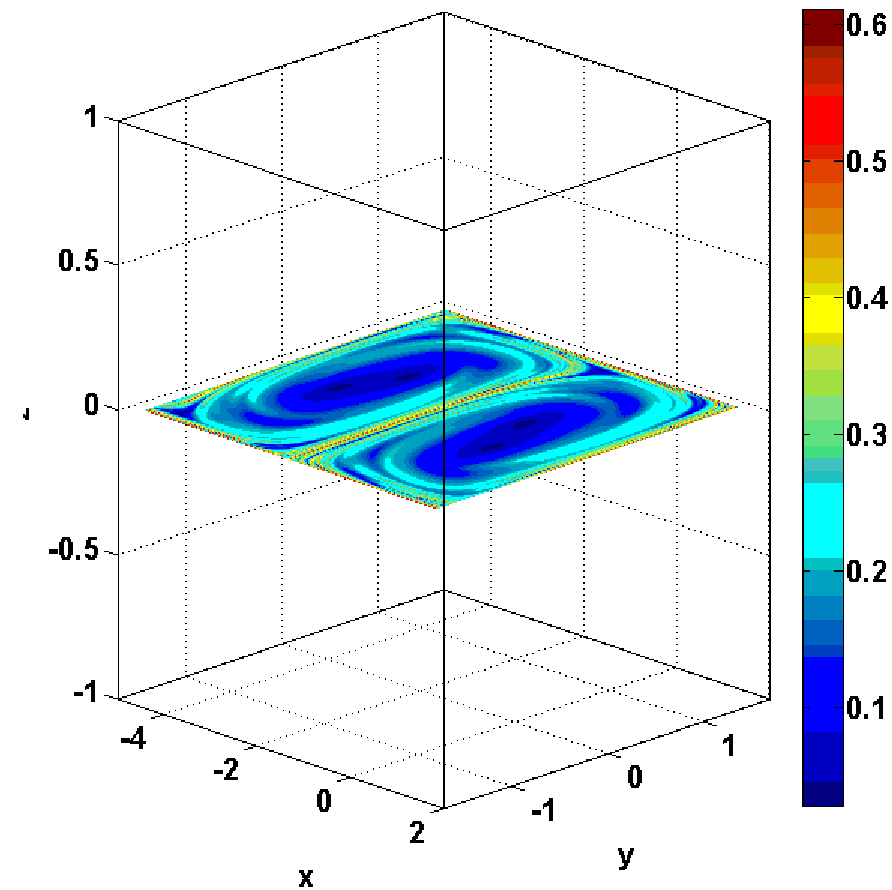
χ 1 Fast Variable

Two Dimensional +Time Dependent Case

- Poincaré Section
- F.T.L.E.

$$\varepsilon_0 = 0.1 \; \varepsilon_1 = 0 \quad \omega = 2.00 \; t = 8\pi/\omega$$





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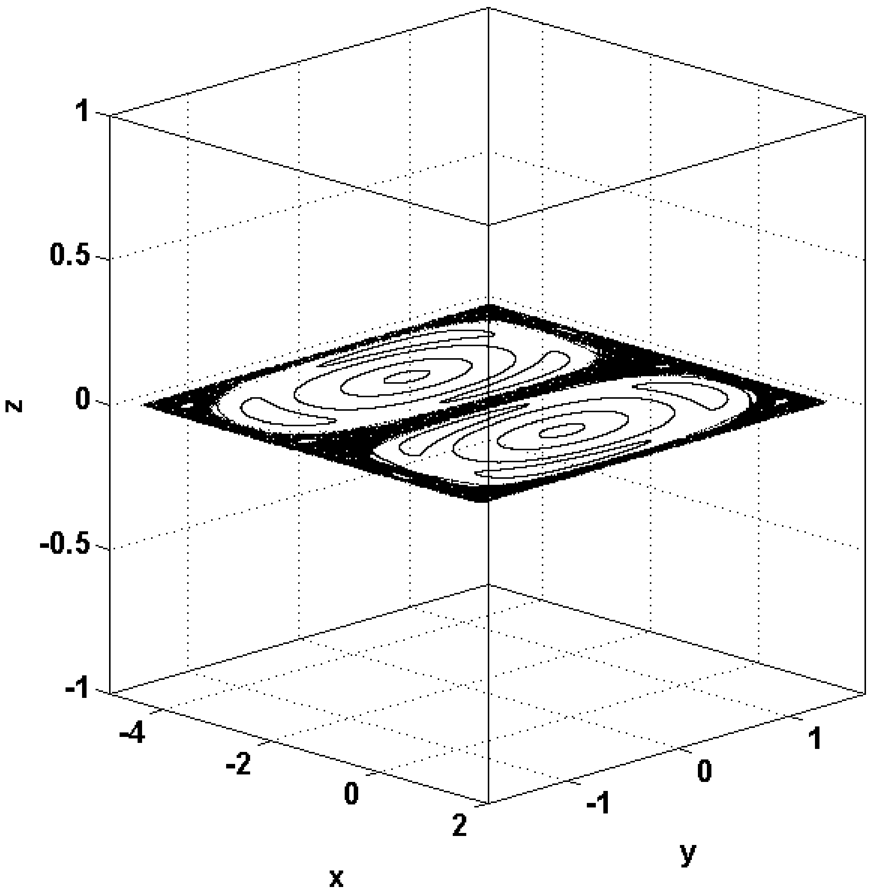
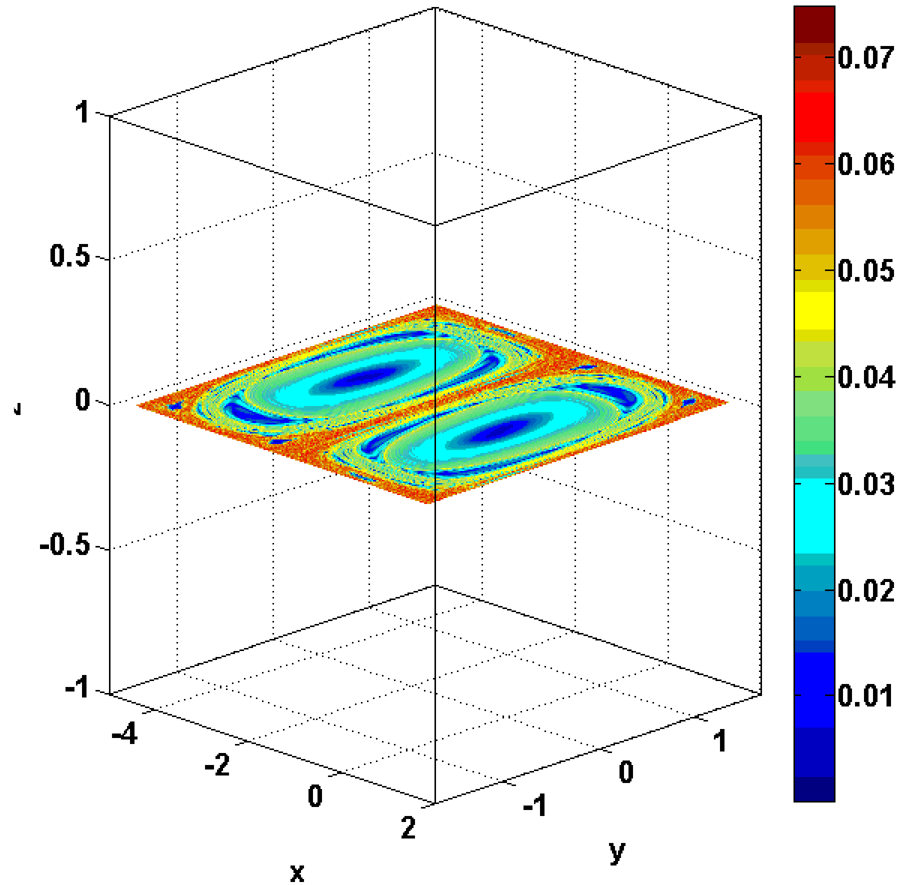
$$\varepsilon_0 = 0.1 \ \varepsilon_1 = 0 \quad \omega = 2.00 \ t = {80\pi}/{\omega}$$





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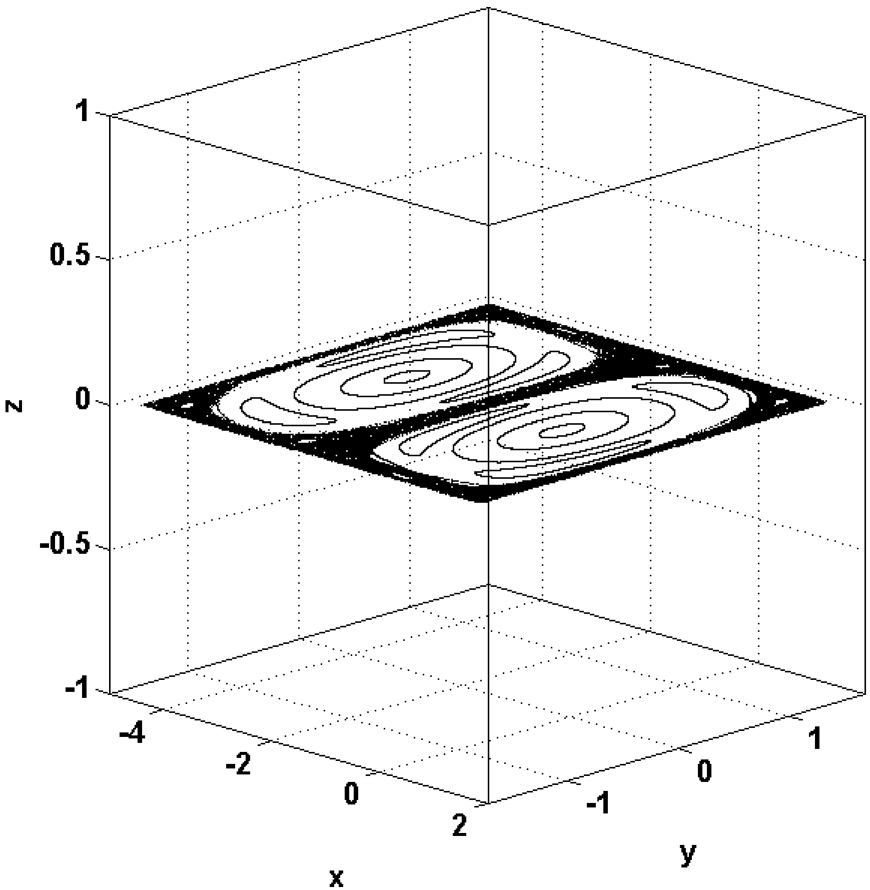

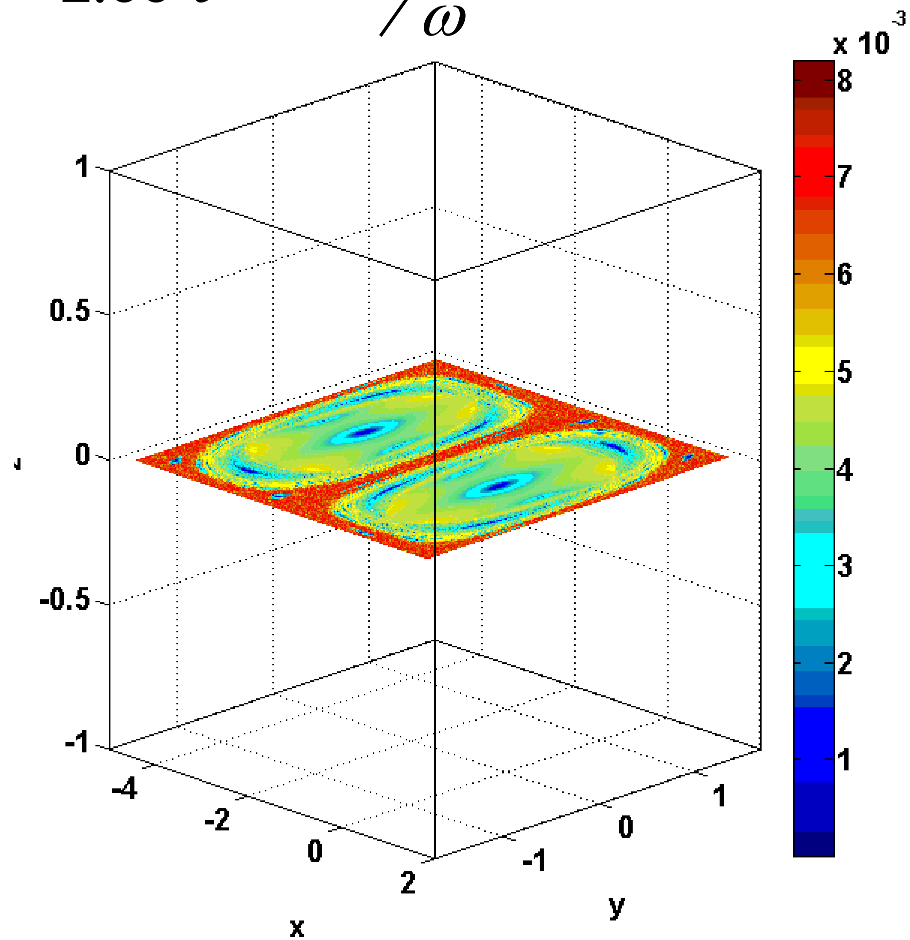

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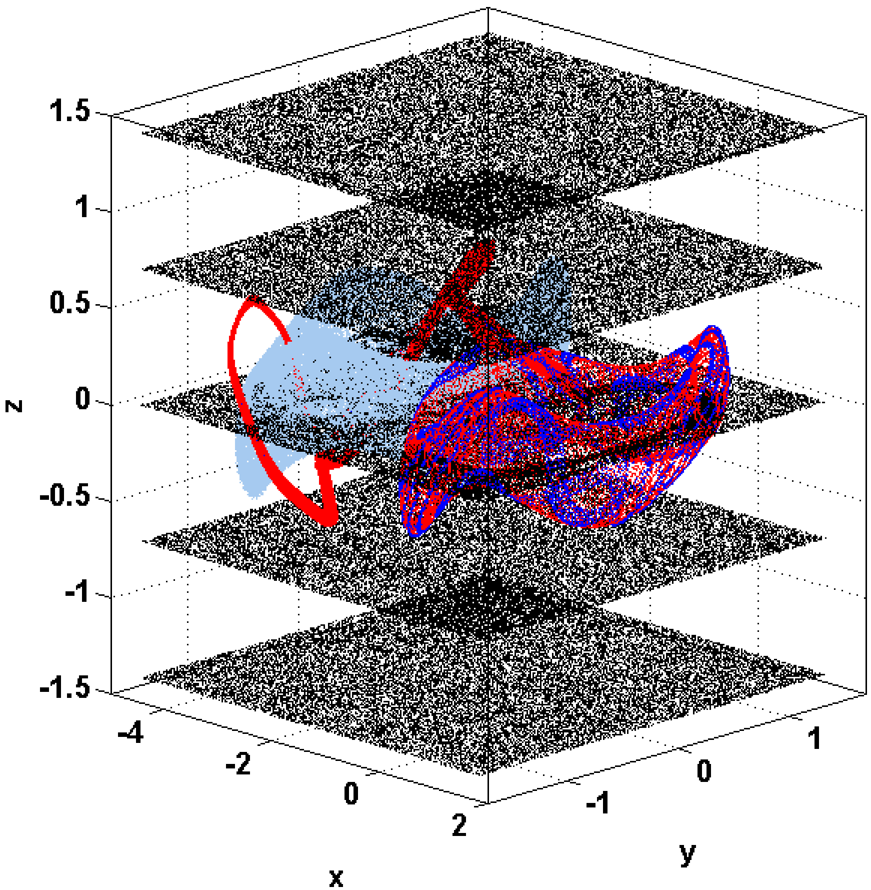

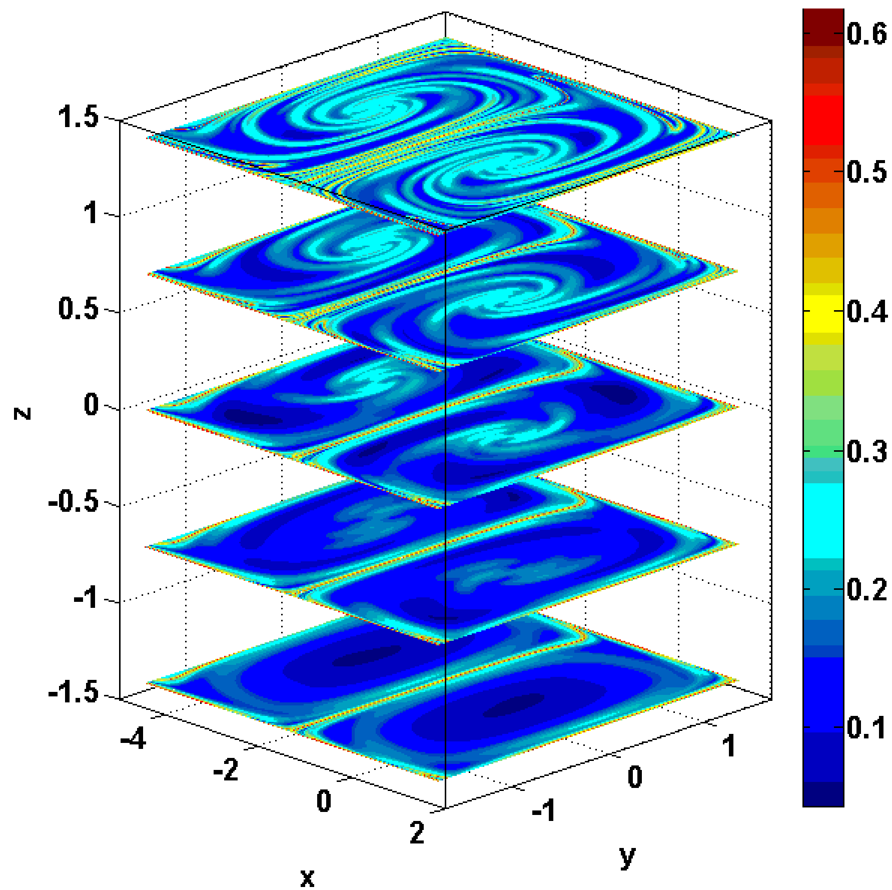

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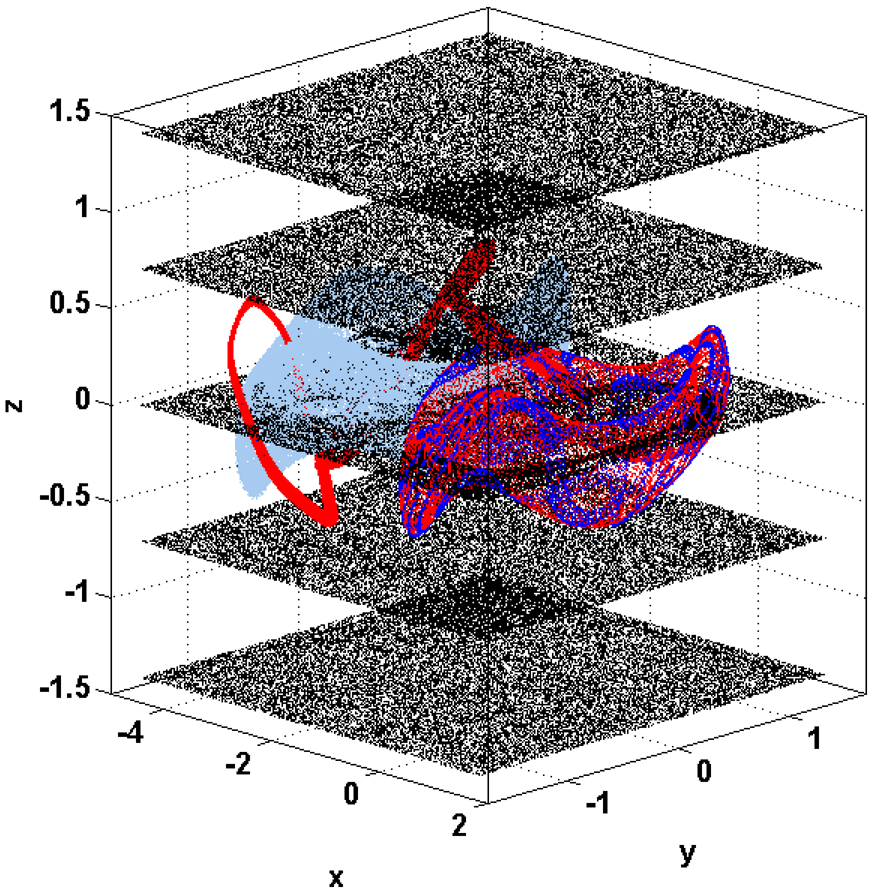

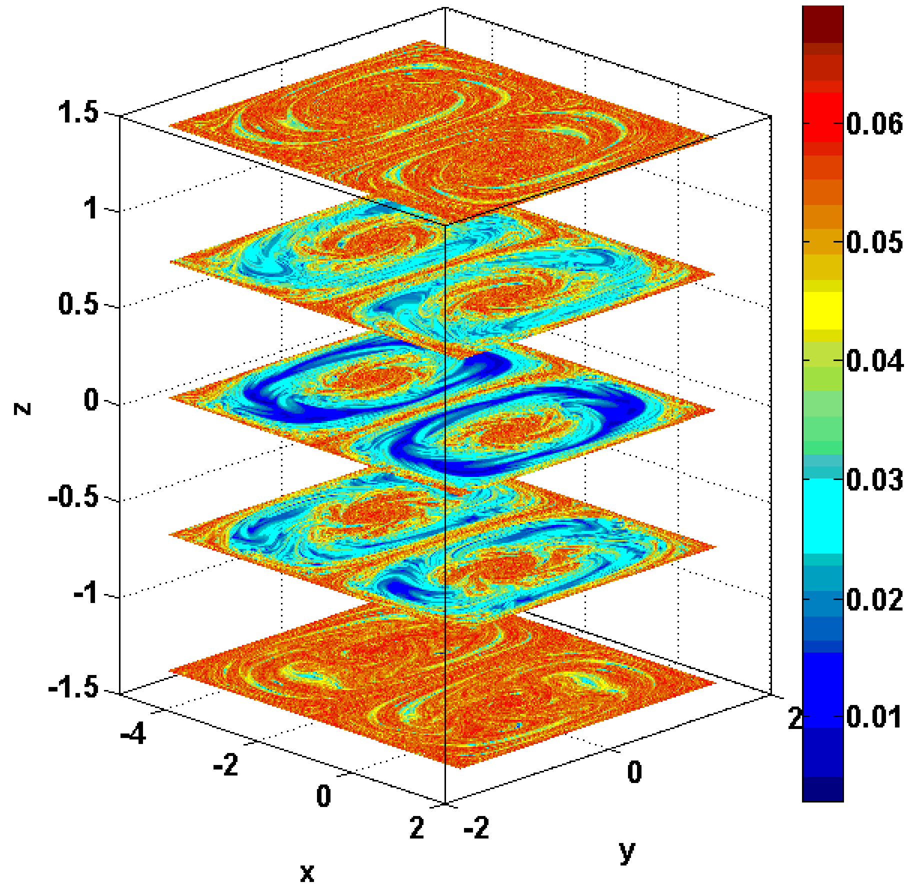

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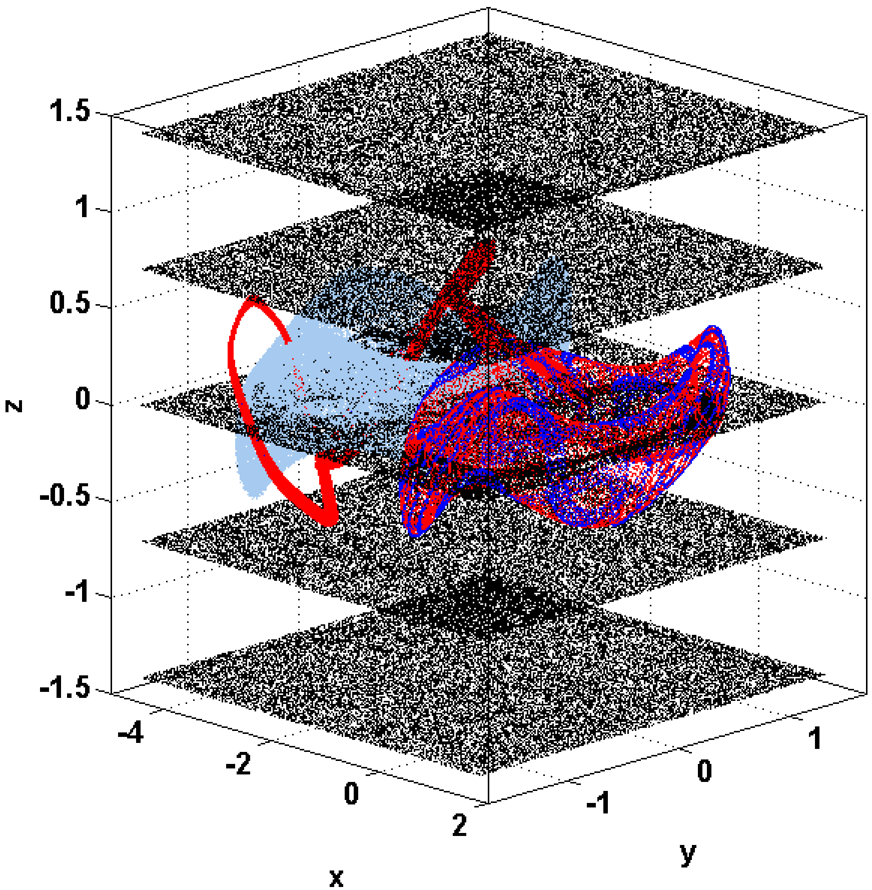

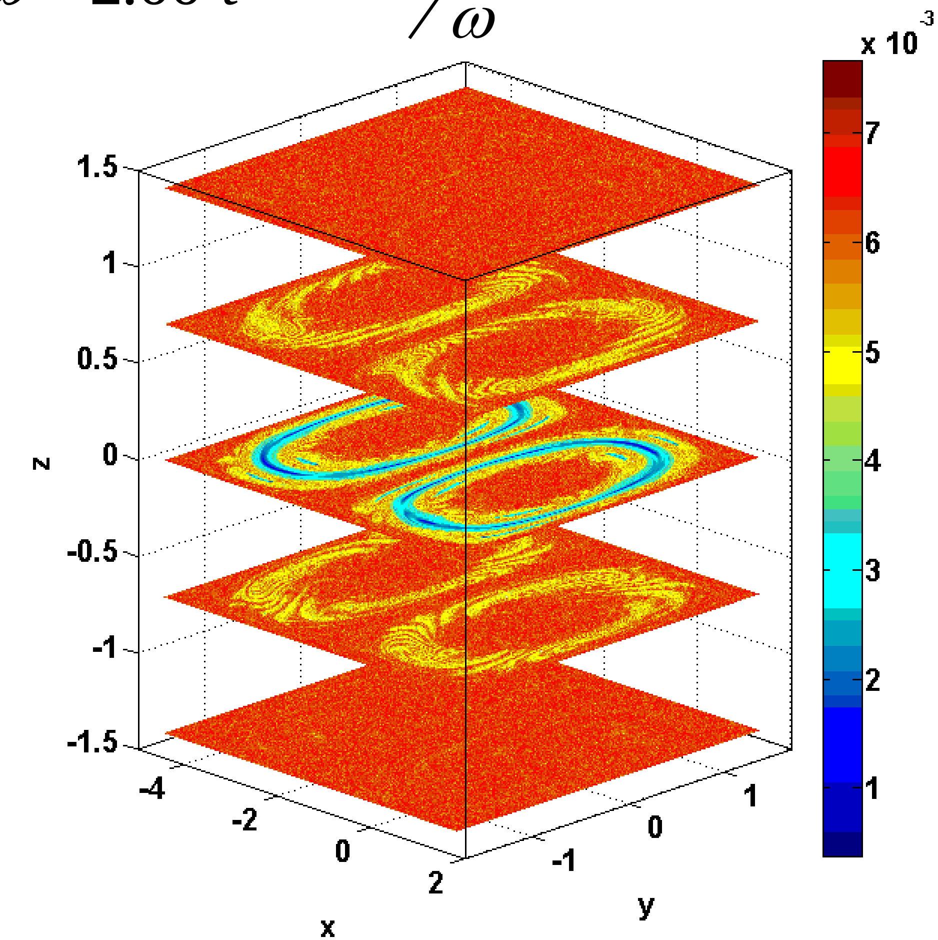

Next Steps

- Resonances
- Capture into Resonance
- Scattering
- Key periodic Orbits