Resonance widths in 3D volume-preserving flows with symmetries subject to a small time-dependent perturbation

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outline

- what is resonance width and why is it important
- formula for resonance width (what sets the width of a resonance)
- comparison between theory and numerical simulations
- time-dependent perturbation

- When a torus breaks up, tori within a certain resonant width are also destroyed

- Resonance width is the half-width of the islands that appear in the resonant layer, as measured in the change ΔI in the action variable from the center to the outer edge of the island: $I_{res} - \Delta I < I_{res} < I_{res} + \Delta I$

- Overlapping resonances lead to the destruction of all surfaces lying between the two resonant tori (Chirikov and Zaslavsky 1972; Chirikov 1972).

- We seek an analytical expression for ΔI

- An expression is known for 2D incompressible flows but not for 3D volume preserving flows

Resonance width: derivation

The trajectory equations for the perturbed flow can be written in action-angle-angle variables:

$$\begin{aligned} \dot{I} &= \epsilon F^0(I,\phi,\theta,\sigma t) \\ \dot{\phi} &= \Omega_\phi(I) + \epsilon F^1(I,\phi,\theta,\sigma t) \\ \dot{\theta} &= \Omega_\theta(I) + \epsilon F^2(I,\phi,\theta,\sigma t), \end{aligned} \tag{1}$$

The perturbation $F^0(I,\phi,\theta,\sigma t)$ can be expanded in a Fourier series

$$F_0(I,\phi,\theta,\sigma t) = \sum_{n,m,l=-\infty}^{\infty} F^0_{nml}(I)\sin(n\phi + m\theta + l\sigma t + \alpha_{nml}). \tag{2}$$

The evolution of I along a trajectory is then given by

$$\dot{I} = \epsilon \sum_{n,m,l=-\infty}^{\infty} F^0_{nml}(I)\sin(n\phi + m\theta + l\sigma t + \alpha_{nml}). \tag{3}$$

For flow in the vicinity of a particular torus $I = I_0 + \delta I$ the phase function

$$
\begin{aligned}
\eta_{nml}(t) = n\phi(t) + m\theta(t) + l\sigma t + \alpha_{nml} &\approx \\
n\phi(0) + m\theta(0) + \alpha_{nml} &+ \\
t\left[n\Omega_\phi(I_0) + m\Omega_\theta(I_0) + l\sigma\right] &+ \qquad \leftarrow O((\delta I)^0) \\
t\left[n\frac{\partial\Omega_\phi}{\partial I}(I_0) + m\frac{\partial\Omega_\theta}{\partial I}(I_0)\right]\delta I &+ \qquad \leftarrow O((\delta I)^1) \\
O((\delta I)^2) + O(\epsilon). &
\end{aligned}
\tag{4}
$$

If $n\Omega_\phi + m\Omega_\theta + l\sigma \neq 0$, η increases linearly in time, the corresponding term is sinusoidal, and δI oscillates but doesn't grow.

However, if $n\Omega_\phi + m\Omega_\theta + l\sigma = 0$, then to the lowest order $\eta \approx n\phi(0) + m\theta(0) + \alpha_{nml}$ is constant and δI grows much faster.

Thus, omitting the oscillating non-resonant terms whose contribution is small, the displacement δI from the resonant torus

$$\dot{\delta I} = \epsilon F^0_{nml}(I_0)\sin(\eta) \tag{5}$$

where

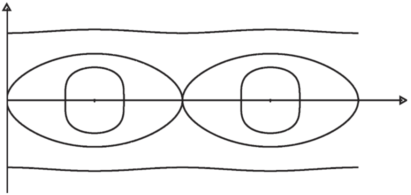
$$\dot{\eta} = [n\frac{\partial \Omega_\phi}{\partial I}(I_0) + m\frac{\partial \Omega_\theta}{\partial I}(I_0)]\delta I. \tag{6}$$

Equations (5) and (6) can be expressed in the Hamiltonian form

$$\dot{\delta I} = -\partial H/\partial \eta \quad \text{and} \quad \dot{\eta} = \partial H/\partial(\delta I) \tag{7}$$

where

$$H = \epsilon F^0_{nml}(I_0)\cos(\eta) + \left[n\frac{\partial \Omega_\phi}{\partial I}(I_0) + m\frac{\partial \Omega_\theta}{\partial I}(I_0)\right]\frac{(\delta I)^2}{2} \tag{8}$$



$$\Delta I = \left(\frac{2\epsilon F^0_{nml}(I_0)}{\left| \frac{\partial}{\partial I} \left[n\Omega_\phi + m\Omega_\theta \right] |_{I=I_0} \right|} \right)^{1/2} \tag{9}$$

with resonance condition $n\Omega_\phi + m\Omega_\theta + l\sigma = 0$

One important difference between the steady and non-steady cases is that for non-steady systems we can have resonances with non-periodic trajectories, for which $\Omega_\phi/\Omega_\theta$ is irrational.

Resonance width expression holds for the **quasi-periodic perturbation**, $F^0(I, \theta, \phi, \sum \sigma_i t)$, with resonance condition replaced by $n\Omega_\phi + m\Omega_\theta + l_i\sigma_i = 0$.

For **degenerate resonances**, when $\partial\Omega_\phi/\partial I(I_0) = \partial\Omega_\theta/\partial I(I_0) = 0$, higher-order terms should be accounted for leading to

$$\Delta I = \left(\frac{\epsilon F^0_{nml}(I_0)(j+1)!}{\left| \frac{\partial^j}{\partial I^j} \left(n\Omega_\phi + m\Omega_\theta \right) \big|_{I=I_0} \right|} \right)^{1/(j+1)} \tag{10}$$

where j corresponds to the first non-vanishing derivative. The corresponding resonance width in frequency is

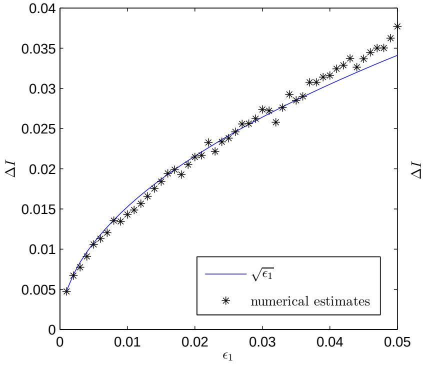
$$\Delta\Omega_{\phi/\theta} = \frac{\partial^j \Omega_{\phi/\theta}}{\partial I^j}(I_0)\frac{(\Delta I)^j}{j!} \propto \epsilon^{j/(j+1)}. \tag{11}$$

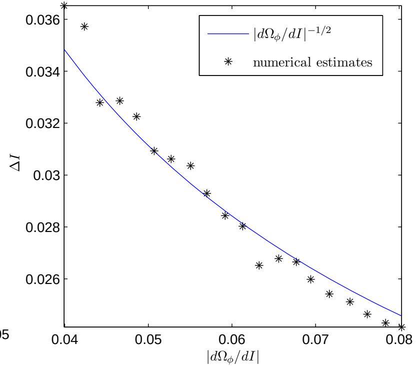
Degenerate resonances ($j > 1$) have smaller resonance widths than nondegenerate tori, so they require larger perturbation strength to overlap leading to **Strong KAM Stability**.

Comparison between theory and numerical simulations

$$\Delta I = \left(\frac{2\epsilon F^0_{nml}(I_0)}{\left| \frac{\partial}{\partial I} \left[n\Omega_\phi + m\Omega_\theta \right] \right|_{I=I_0} \right|} \right)^{1/2}$$

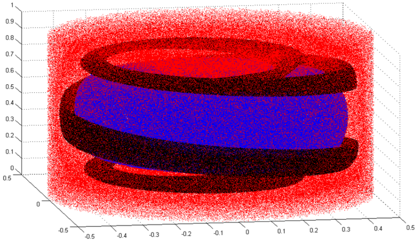
In the rotating can model, $\partial\Omega_\theta/\partial I = 0$ so $\Delta I \propto \sqrt{\epsilon / \left| \frac{\partial \Omega_\phi}{\partial I}(I_0) \right|}$.



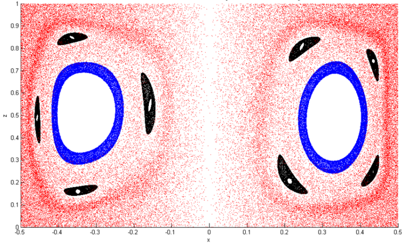


Perturbation with periodic time-dependence

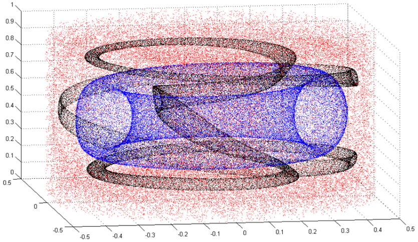
trajectories



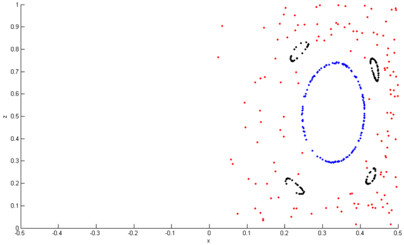
Poincare' map in θ only



Snap shot of time-dependent tori

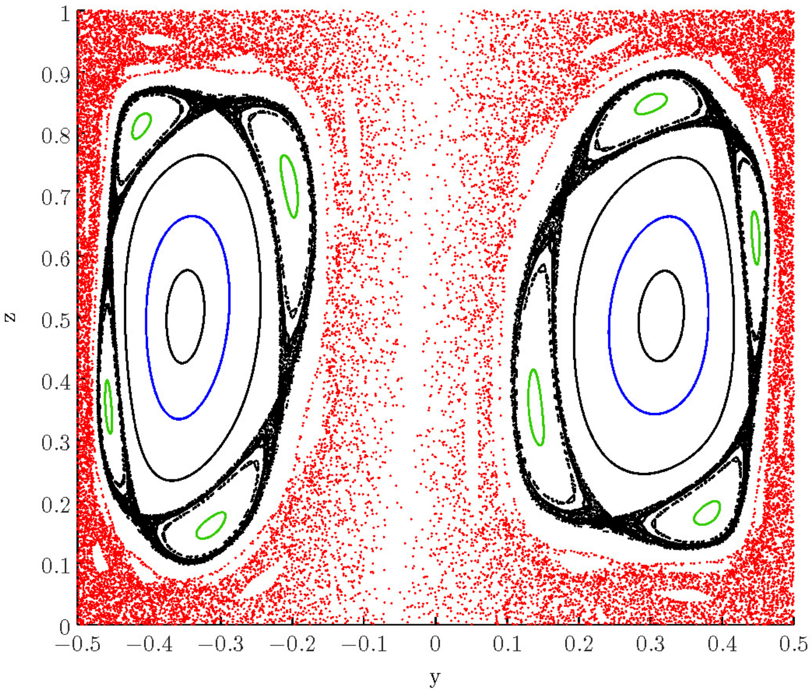


Double Poincare' map



Summary

- Resonance widths are important as they control the extent of chaos
- We derived formula for resonance widths (applies to both steady and nonsteady cases)
- And compared theory with numerical simulations
- Strong KAM stability predicts robust transport barriers near degenerate tori
- Time-dependent case under investigation



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
z
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
y