Complexity – measure – based method (CM method) for identifying LCSs

Trajectories of fluid parcels in 3d time-dependent flows can be very complex. **Our goal is to identify, among all these convoluted motions, clusters of trajectories with similar Lagrangian behavior.** This will help to locate coherent water masses traveling together as the flow evolves. **Boundaries between the clusters correspond to LCSs** and act as transport barriers with very little fluid exchange across them.

I. Measuring trajectory complexity

In time-dependent fluid flows, Lagrangian trajectories of fluid particles have very different behavior – from simple stationary points to complex chaotic trajectories. These differences in trajectory behavior can be quantified in a number of ways.

The first, easiest and most intuitive, measure of trajectory complexity is **trajectory arclength (or pathlength)** – the length of a path of a fluid parcel over some time interval T,

$L = \int_{t_0}^{t_0+T} |\dot{x}(t)| dt$, or, if trajectory is sampled with discrete time steps dt, $L = \sum_i \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$.

L is zero for stationary points and increases with increasing complexity.

The second convenient measure of trajectory complexity is its **correlation dimension**, which is closely related to the fractal dimension. Loosely speaking, correlation dimension C measures area (volume in 3d) occupied by a trajectory. C can vary from 0 for simplest stationary points to 2 (3 for 3d flows) for chaotic trajectories that densely cover area (volume for 3d flows). The correlation dimension can be easily computed using the box counting algorithm: cover the domain with adjucent squares of length s and count the number of trajectory points, N_j, inside each square (Fig. 1), then repeat the counting for smaller and smaller s (more and more squares), and estimate $C = \log(\frac{\sum N_j^{\,2}(s)}{N^2}) / \log(s)$.

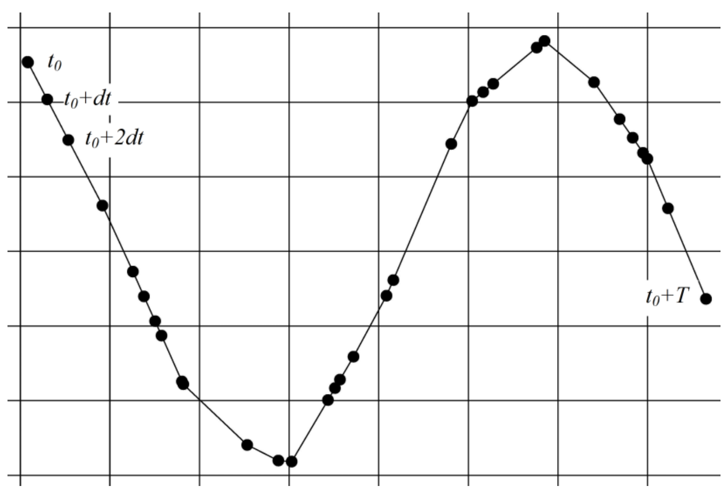

Figure 1: Box counting algorithm: cover domain with adjacent squares of length s and count the number of trajectory samples (dots), Nj, inside each square.

Alternatively, one can use **ergodicity defect** (or ergodicity deficit), *d*, to measure trajectory complexity. Ergodic hypothesis is that the space average of any physical quantity is equal to the time average of same quantity along trajectory. Ergodic trajectory thus should uniformly cover the domain and spend equal amount of time in each little part of the domain. The most non-ergodic case is a stationary trajectory (stationary point), which spends all time at one location and has $d=1$. The most ergodic case is a chaotic trajectory that covers the whole domain and has $d=0$. Similar to *C*, *d* can also be estimated from the box counting as $d=\sum\left(N_j(s)/N-s^2\right)^2$.

II. Basic principles of the CM method

We want to separate the domain into regions with qualitatively different Lagrangian properties, where trajectories have qualitatively different behavior. In other words, **we want to identify, among all trajectories, clusters with similar complexities. We can then locate LCSs as the boundaries between different regions**, i.e., as curves across which we have largest gradient of complexity**.**

This method allows separating a Lagrangian core of an eddy from its surroundings, and separating trajectories lying on opposite sides of a manifold (Fig. 2). Stable manifolds are revealed by forward integration, unstable manifolds – by backward integration of trajectories.

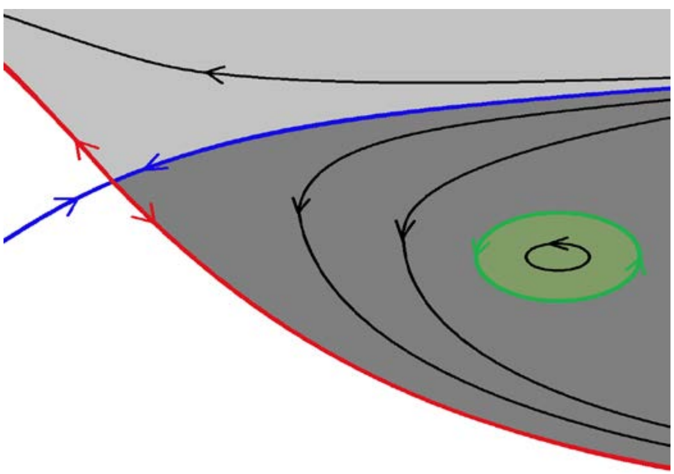

Figure 2: Flow near the hyperbolic and elliptic regions. Blue/red are stable/unstable manifolds of a hyperbolic trajectory, green is the boundary of an elliptic region (eddy core). Trajectories on opposite sides of a stable manifold diverge from each other in forward time. Trajectories inside the eddy core stay within the eddy. These three classes of trajectories have qualitatively different complexities resulting in high complexity gradient across the blue and green curves.

III. Application of the CM method

The computation starts with **seeding a large number of simulated fluid parcels within the domain of interest and evaluating their trajectories**. Once trajectories are computed, the second step is to estimate trajectory complexities, i.e., to **compute *L*, *C* or *d* for each trajectory**. Color-coded complexity maps, showing *L*, *C* or *d* as function of trajectory's initial position will then **reveal regions with similar complexity** (uniform color) **separated by LCSs –** curves of most noticeable color change (Fig. 3).

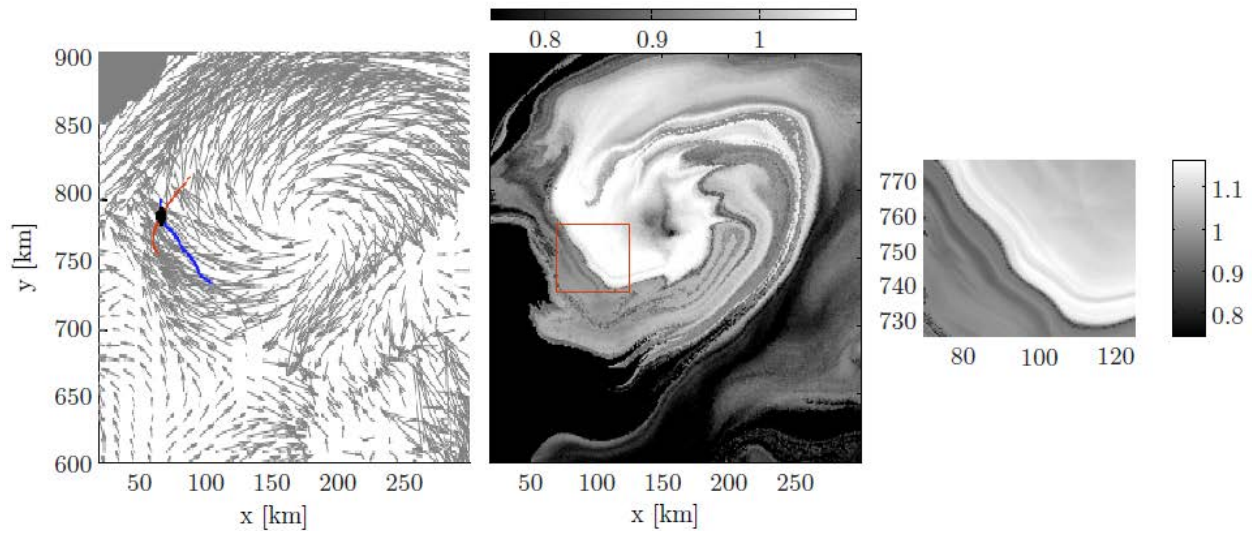

Figure 3: (left) Velocity and manifolds; (middle and right) *C* field for the numerically-generated flow field produced by ROMS. Several LCSs (curves of large color change) are seen at the perimeter of the eddy. The core region is not well-defined suggesting that this eddy is exchanging fluid with its surroundings over the integration period of 2 weeks.

A recurring theme for all Lagrangian methods is how long to integrate trajectories. The answer depends on which temporal and spatial scales one is interest in. For example, if one is interested in mesoscale eddies, T_{int} should be comparable to a lifetime of an eddy. If, on the other hand, one is interested in basin-wide circulation over a decade, T_{int} should be set to a decade. In the latter case, one should not expect to identify LCSs associated with mesoscale features whose lifetime is $<< T_{\mathrm{int}}$.

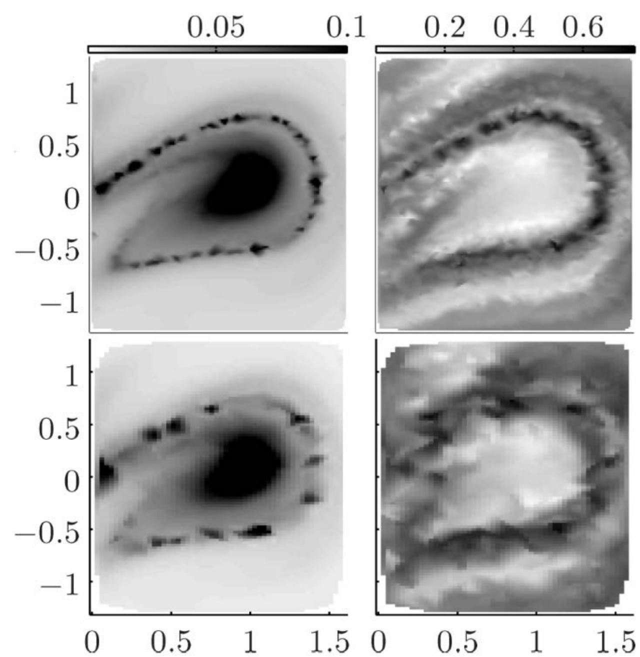

Figure 4: Comparison between CM (left) and FTLE (right) fields computed using (top) 2550 and (bottom) 640 randomly seeded simulated drifters advected by the Duffing Oscillator flow.

One advantage of the CM method over the Lyapunov-exponent-based methods is its ability to work with nonuniformly-spaced trajectories. Unlike LE methods, which use separation rates between particles and produce noisy fields in these settings, CM method uses complexities of individual trajectories and produces much cleaner images of LCSs (Fig. 4).

More details on the CM method can be found in
Rypina, I. I.; Scott, S. E.; Pratt, L. J. & Brown, M. G. Investigating the connection between complexity of isolated trajectories and Lagrangian coherent structures *Nonlin. Proc. Geophys.*, **2011**, *18*, 977-987, doi:10.5194