Ocean Flow Models

- Understanding ocean flow and mixing can help predict trajectories of other objects carried passively in the ocean (applicable to AUV's).
- The ocean is a thin fluid envelope spread on a rapidly rotating Earth. Dynamics are largely two-dimensional with weak vertical variation.
- Behavior is governed by the dimensionless Rossby number $Ro = U/Lf$, ratio of local vorticity to planetary vorticity.

Quasigeostrophic (QG) Equations

These are equations valid at large scales in the ocean. The equations of motion are reduced for $Ro \ll 1$:

ζ	Vorticity
$\theta = f\partial_z\psi$	Buoyancy ($\sim$ density)
ψ	Streamfunction

bulk $\quad \partial_t\zeta = -J(\psi,\zeta) + f\partial_z w$

surface $\quad \partial_t\theta = -J(\psi,\theta) - N^2 w$

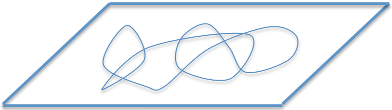
bulk $\quad q = \left[\partial_{xx} + \partial_{yy} + \partial_z\left(\frac{f}{N}\right)^2\partial_z\right]\psi$

Velocities are then given by $(u, v, w) = \left(-\partial_y\psi, \partial_x\psi, w\right)$.

Pedlosky, 1982

Surface Quasigeostrophic Equations

Surface QG (SQG) assumes potential vorticity $q = 0$, so the flow is determined by the boundary temperature distribution



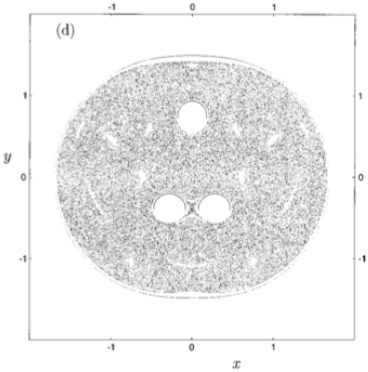
Some $\theta = f\,\frac{\partial \psi}{\partial z}$ distribution on the surface

Motion induced below by q=0

If we rescale our variables, we find are solving $\nabla^2 \psi = 0$ in the volume subject to $\partial_z \psi = \theta$ on the boundary.

Understanding Vortex Behavior

Three point vortices in 2D (which have regular motion) induce chaotic motion in passive scalars. We can quantify this chaos and thus compare the transport properties of SQG point vortices to those of the 2D system.



Kuznetsov and Zaslavsky, 1998

Equations of Motion on the Surface

We specify a temperature distribution on the surface:

$$\theta_0 = \sum_n \kappa_n \delta(x - x_n)\delta(y - y_n).$$

We find the resulting vortex velocities:

$$(\dot{x}_n, \dot{y}_n) = \sum_{m \neq n} \frac{\kappa_m}{2\pi} \frac{1}{|\vec{x}_n - \vec{x}_m|^3}(-y_n + y_m, x_n - x_m).$$

The velocities in the interior are then found by

$$\psi_0 = \frac{1}{2\pi|\vec{x} - \vec{x}_n|},$$

$$\vec{u}_0 = (-\partial_y \psi_0, \partial_x \psi_0) = \sum_n \frac{\kappa_n}{2\pi|\vec{x} - \vec{x}_n|^3} (-y + y_n, x - x_n, 0) \, .$$

First Order Expansion

We wish to determine the vertical velocity.

Let $\epsilon = Ro$.

$$
\begin{aligned}
u &\sim -\frac{\partial \psi_0}{\partial y} - \epsilon \left(\frac{\partial \psi_1}{\partial y} + \frac{\partial F_1}{\partial z} \right), \\
v &\sim \frac{\partial \psi_0}{\partial x} + \epsilon \left(\frac{\partial \psi_1}{\partial x} - \frac{\partial G_1}{\partial z} \right), \\
w &\sim \epsilon \left(\frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} \right) = -\frac{D_0 \theta_0}{Dt}.
\end{aligned}
$$

F, G are curl potentials of the velocity.

Muraki *et al*, 1999

3D Transport

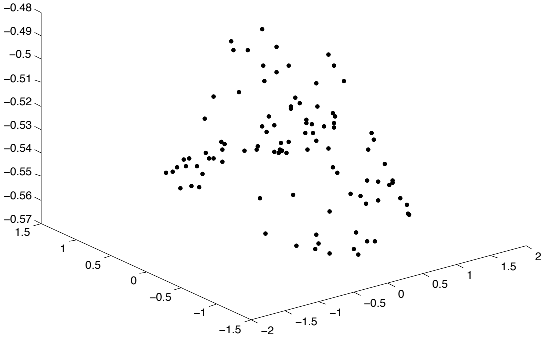
- For vortices we first take w_1 but not u_1, v_1:

$$w_1 = 3z \sum_n \frac{\kappa_n}{2\pi} \frac{\left(\vec{u}_0 - \dot{\vec{x}}_n\right) \cdot (\vec{x} - \vec{x}_n)}{|\vec{x} - \vec{x}_n|^5}.$$

- With this we can observe 3D particle trajectories and consider their transport properties.
- If we strobe the position of the particle at every period of vortex motion, we obtain what is called a Poincaré plot.

Poincaré Plot in 3D

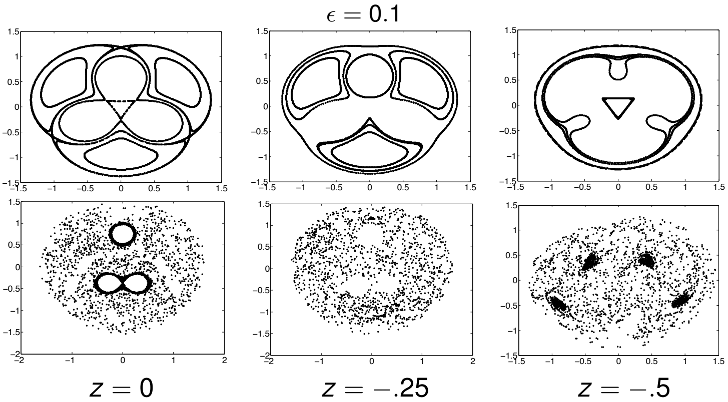
An example for $\epsilon = 0.1$:



But this is hard to compare to other plots, so we project into 2D.

Poincaré 2D Projections at Various Heights

- The top row is for a vortex configuration that produces regular motion. Each curve represents a particle trajectory.
- The bottom row is for a different vortex configuration. The cloud of spots indicates good mixing in that flow.
- Each column represents particles initialized at different heights.



Conclusion

- SQG produces an interesting system with 2D dynamics and 3D transport.
- We can use SQG to examine the robustness of 2D models.
- We see with Poincaré sections that there is clear variation in particle trajectories from different heights.
  - We do not know whether this is caused only by vertical variation of horizontal velocity or whether vertical velocity is also affecting the particle paths.
- We can explore other surface distributions to further explore these effects.

References

Pedlosky, J., 1982. "Geophysical fluid dynamics." *New York and Berlin, Springer-Verlag*.

Kuznetsov, L. and G. M. Zaslavsky, 1998: Regular and chaotic advection in the flow field of a three-vortex system. *Phys. Rev. E*, **58** (6), 7330-7349.

Muraki, D. J., C. Snyder, R. Rotunno, 1999: The next-order corrections to quasigeostrophic theory. *J. Atmos. Sci.*, **56**, 1547-1560.