Leaving Flatland: Aspects of 3D Dynamical Systems Diagnostics in Simple GFD Flows

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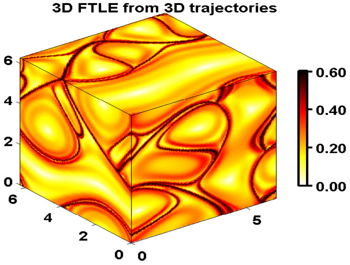
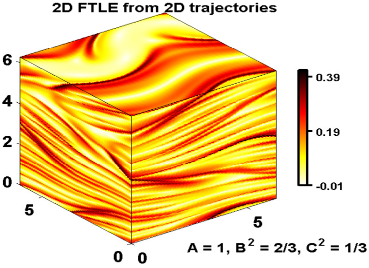
Lagrangian Coherent Structures (LCSs)

- LCSs are identified as the local maxima of the Finite-Time Lyapunov Exponent (FTLE) fields computed from grided trajectories.
- In ocean models, very little is known about LCSs in (3D+time):
  1. The vertical velocity is diagnostic
  2. The high cost of the computations involved in the 3D trajectories.
- Branicki & Kirwan, 2010 (IJES): Stitch together 2D FTLE fields calculated from 2D velocity data to form the 3D geometry.





Lagrangian Coherent Structures (LCSs)

- Consider the fluid dynamics flow

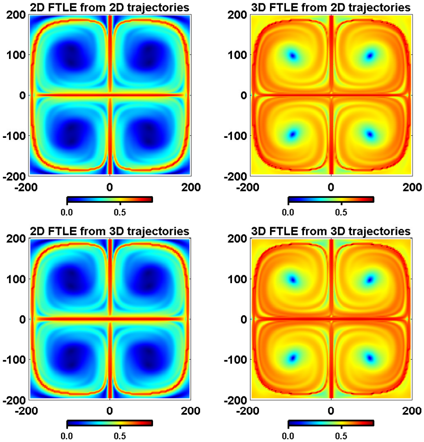
$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t), \ \mathbf{x} \in \Omega \subset \mathbb{R}^n, \ t \in [0, T]$$

- The FTLE over time $[0, T]$ is

$$\Lambda = \frac{1}{2T} \log \left[\lambda_{\max} \left((\nabla \mathbf{x})^* \nabla \mathbf{x}\right)\right],$$

$$\nabla \mathbf{x} = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix}$$

$$S_V = \sqrt{u_z^2 + v_z^2}, \ S_H = \sqrt{w_x^2 + w_y^2}$$



3D Steady Quadrapole-type Flows

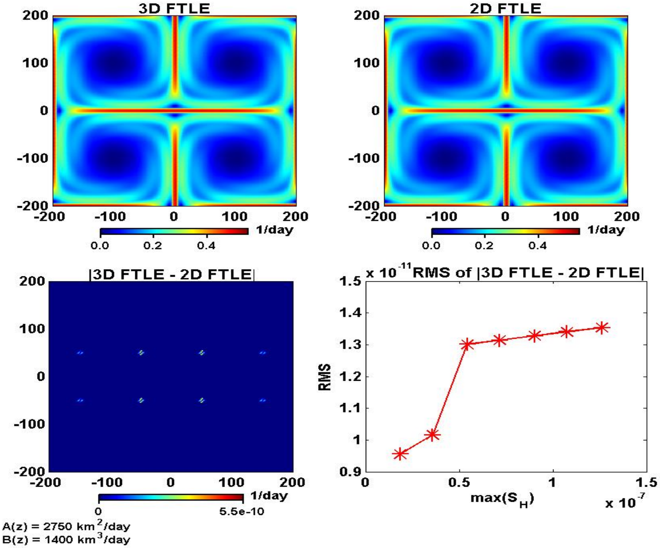
- Consider a simple 3D GFD flow

$$u = -\pi \left(\frac{A(z)}{L_y} + \frac{B_z}{L_x} \right) \sin\left(\frac{\pi x}{L_x}\right) \cos\left(\frac{\pi y}{L_y}\right)$$
$$v = \pi \left(\frac{A(z)}{L_x} - \frac{B_z}{L_y} \right) \cos\left(\frac{\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_y}\right)$$
$$w = \pi^2 B(z) \left(\frac{1}{L_x^2} + \frac{1}{L_y^2} \right) \cos\left(\frac{\pi x}{L_x}\right) \cos\left(\frac{\pi y}{L_y}\right)$$

  $x, y, z \in [-L_x, L_x] \times [-L_y, L_y] \times [-H, 0]$.
- $A(z)$ and $B(z)$ are chosen to study the following three main cases
  1. Vertical velocity with no vertical shear
  2. Vertical velocity with a small shear
  3. Vertical shear with no vertical velocity

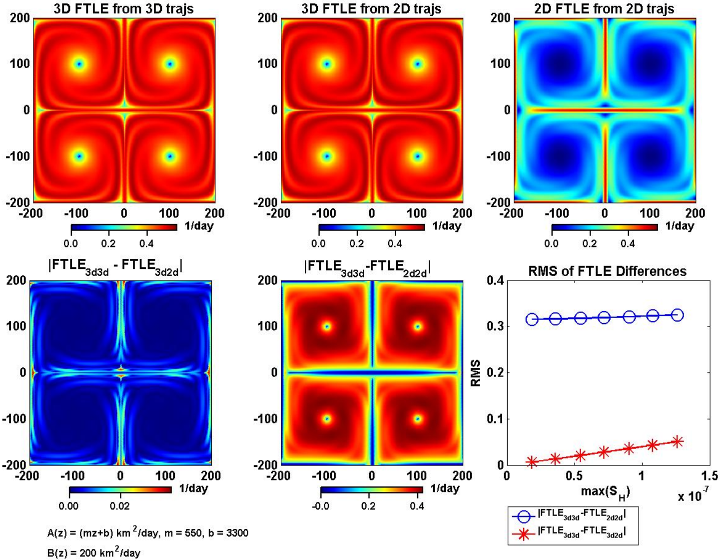
Vertical Velocity with no Vertical Shear

- $B(z) \neq 0$ and $A(z) = \text{const.}, \forall z.$



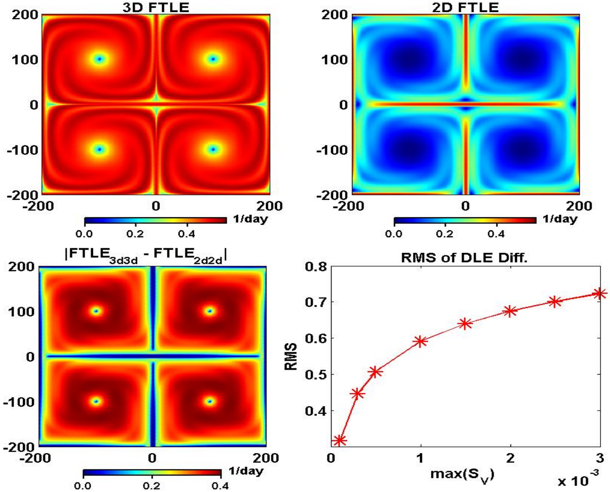
Vertical Velocity with a Small Shear

- $B(z) \neq 0$ and $A(z) = mz + b$, $\forall z$.



Vertical Shear with no Vertical Velocity

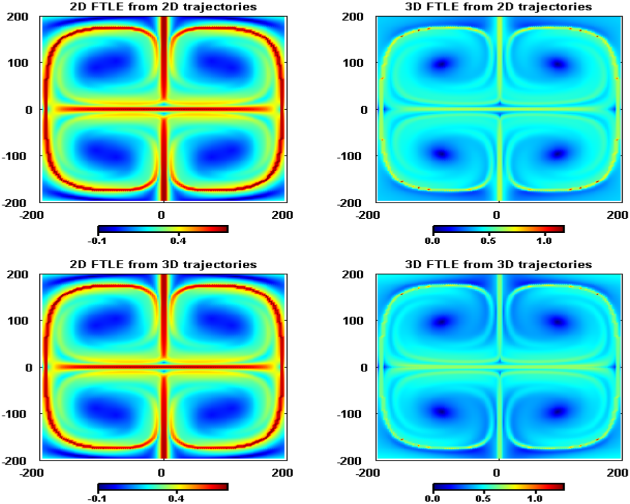
- $A(z) = mz + b$, and $B(z) = 0\ \forall z$.



A(z) = (mz+b) km^2/day, m = 550, b = 3300
B(z) = 0 km^2/day

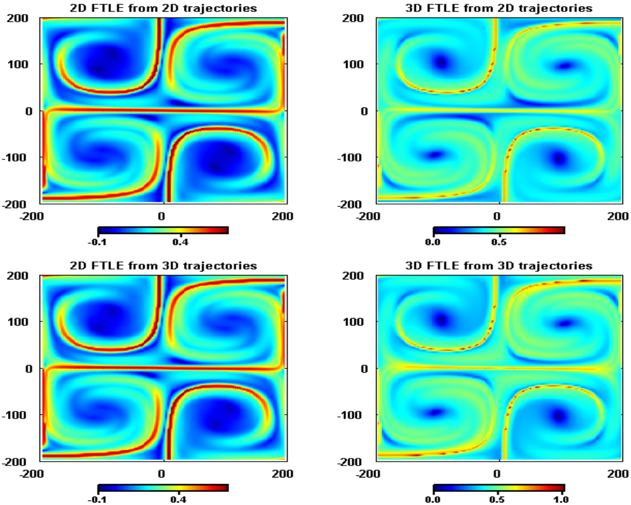
3D Time Dependent Quadrapole Flows

- The time is defined to introduce pure normal deformation to the steady quadrapole.



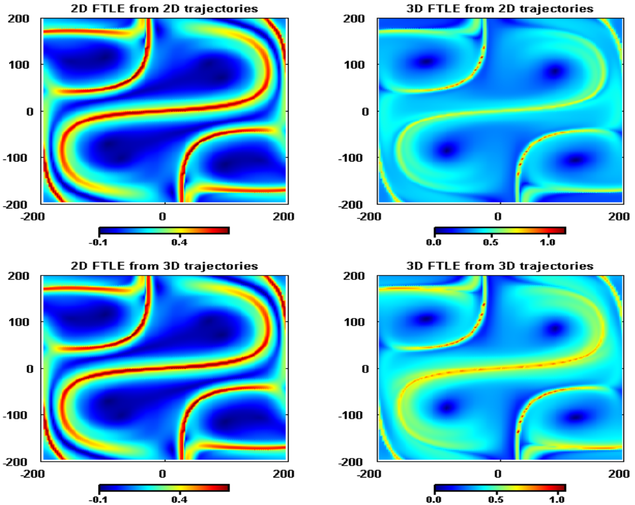
3D Time Dependent Quadrapole-type Flows

- The time is defined to introduce pure shear deformation to the steady quadrapole.

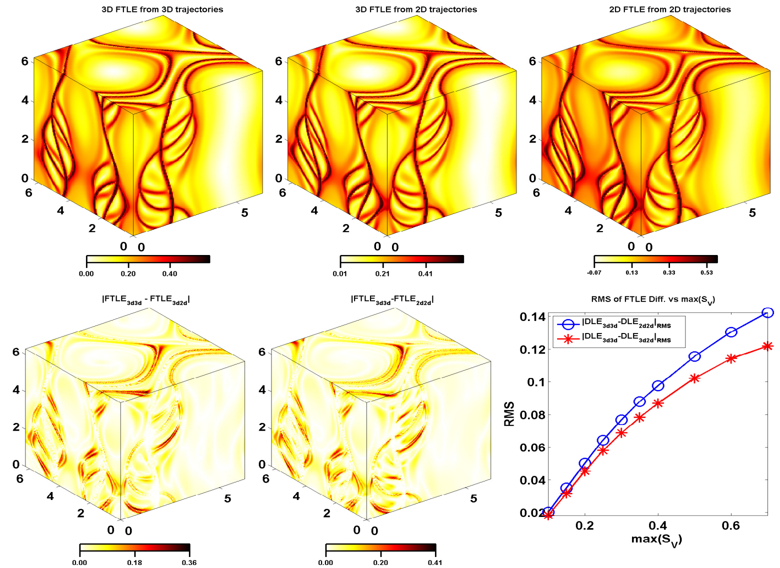


3D Time Dependent Quadrapole-type Flows

- The time is defined to introduce rotation to the steady quadrapole.

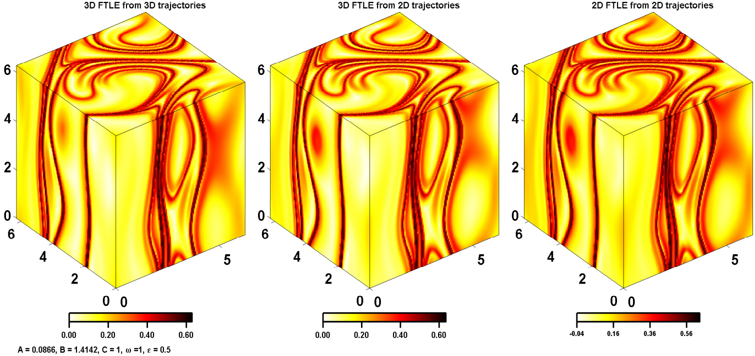


3D Steady ABC flows



3D Time-Dependent ABC Flows

$$u = A\sin(z + \epsilon\sin(\omega t)) + B\cos(y + \epsilon\sin(\omega t)),$$
$$v = B\sin(x + \epsilon\sin(\omega t)) + A\cos(z + \epsilon\sin(\omega t)),$$
$$w = C\sin(z + \epsilon\sin(\omega t)) + B\cos(x + \epsilon\sin(\omega t))\ x, y, z \in [0, 2\pi]^3,$$



Conclusions and Remarks

1. The vertical shear plays a fundamental role in the difference in the FLE fields computed from 3D and 2D-Stitched trajectories.
2. Calculation of 3D DLE from 3D trajectories becomes important ONLY IF the vertical shear is large.
3. The vertical velocity effect in the FTLE fields difference is negligible, in particular when the vertical shear is very small.
4. In Ocean models, accurate measurements of the vertical velocity doesnt exist. However, computing 3D DLE computed from 2D-Stitched trajectories is a good approximation of the 3D FTLE field obtained from 3D trajectories.
5. The difference between 3D FTLE (computed from 3D or 2D-Stitching trajectories) and 2D DLE (computed from 2D trajectories) becomes significant in the presence of the vertical shear.