# MESOCHRONIC ANALYSIS FOR 3D FLOWS

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ONR MURI Ocean 3D+1
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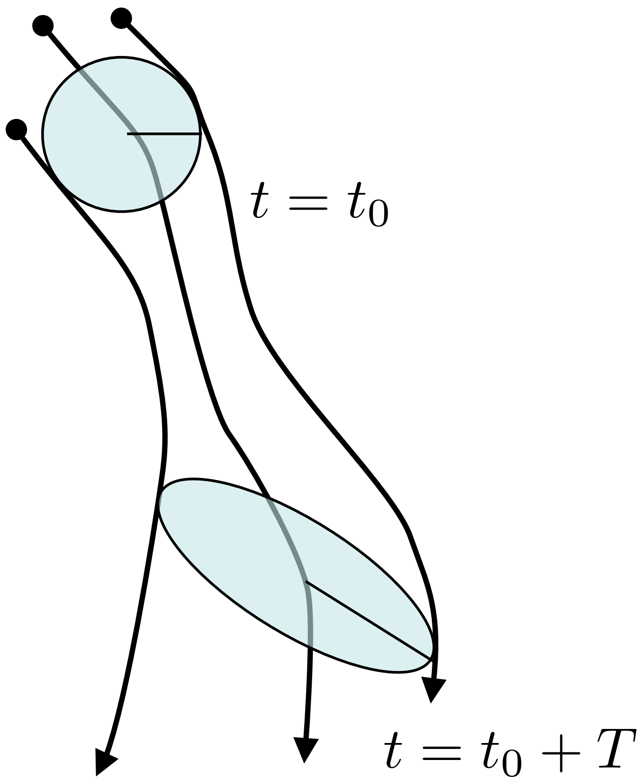
# Linear deformation of the material is the basis for several flow analysis techniques.




## Maximal Finite-Time Lyapunov Exponents:

Detect only the maximal **magnitude** of deformation, **not its character** (e.g., shear, rotation).
Analysis often depends on locally maximising curves as barriers to transport.

## Mesochronic Analysis:

Classifes deformation based on its **character**, not its magnitude.

## Our focus is on the Flow Map:

$$\dot{x}_p = f(t, x_p), \;\; x_p(0) = p$$

Flow map assigns final position to initial.

$$p \longrightarrow \Phi_0^T(p) = x_p(T)$$

$$T \to 0^+$$

Instantaneous analysis is equivalent to vector field analysis.

In unsteady flows, instantaneous analysis gives poor predictions. Move to finite times.

$$T > 0$$

# Deformation by the Flow Map is captured by the Jacobian of trajectory averages of the velocity field.

**Flow map can be interpreted as a Lagrangian average of the velocity field.**

Flow map $$\Phi(p,T) = p + \boxed{\int_0^T f(\tau, x_p(\tau))d\tau}$$

Average Lagrangian velocity $$\tilde{f}(p,T) = \frac{1}{T}\boxed{\int_0^T f(\tau, x_p(\tau))d\tau}$$

$$\Phi(p,T) = p + T\tilde{f}(p,T)$$

**Mesochronic Jacobian captures the linear deformation by the flow.**

$$J_{\tilde{f}}(p,T) = \frac{J_\Phi(p,T) - \mathrm{Id}}{T} = \begin{bmatrix} \partial_1 \tilde{f}_1(p,T) & \partial_2 \tilde{f}_1(p,T) & \dots \\ \partial_1 \tilde{f}_2(p,T) & \partial_2 \tilde{f}_2(p,T) & \\ \vdots & & \ddots \end{bmatrix}$$

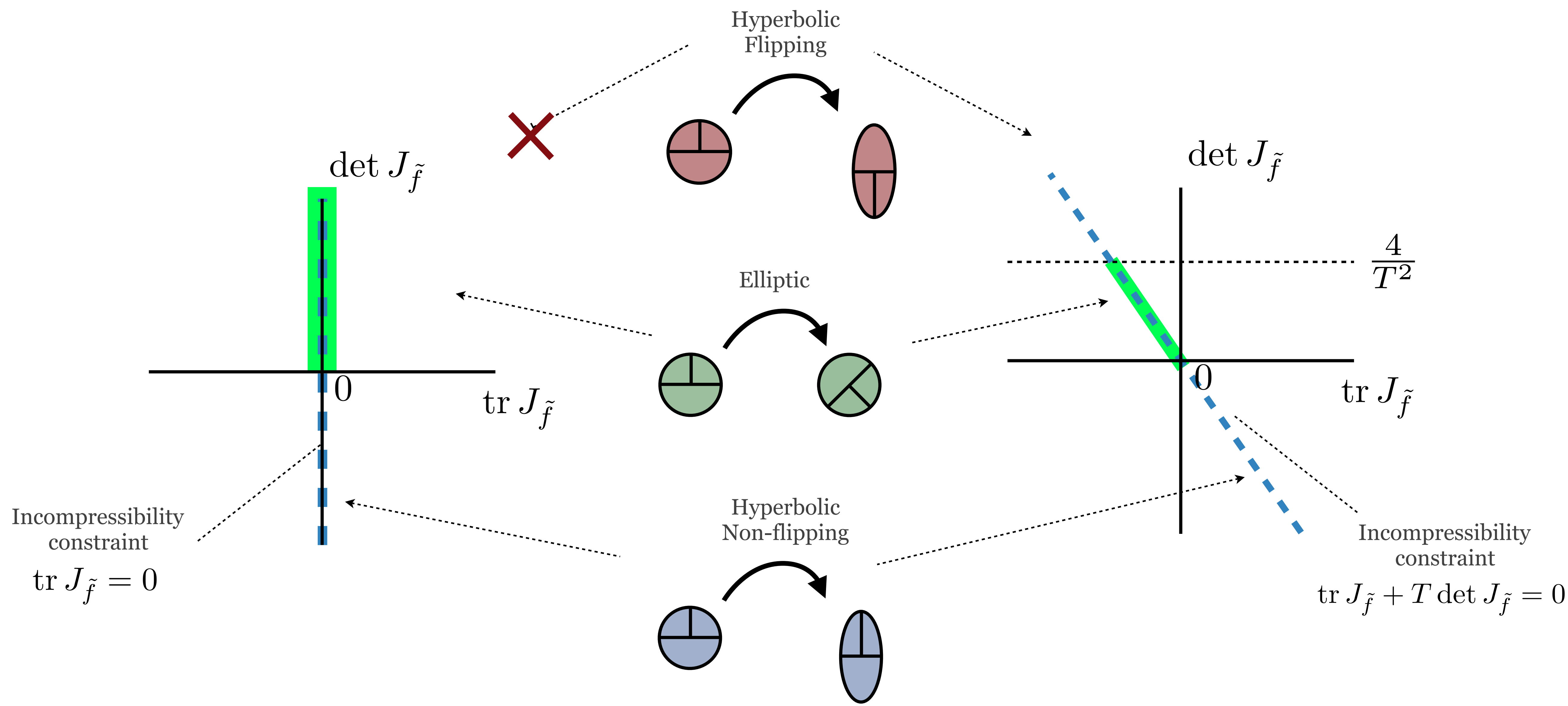
**Locations of the eigenvalues determine the character of deformation.**

# In 2D incompressible flows, a single quantity captures the deformation character.

**Okubo-Weiss:** $T = 0^+$

**Mesochronic Analysis:** $T > 0$

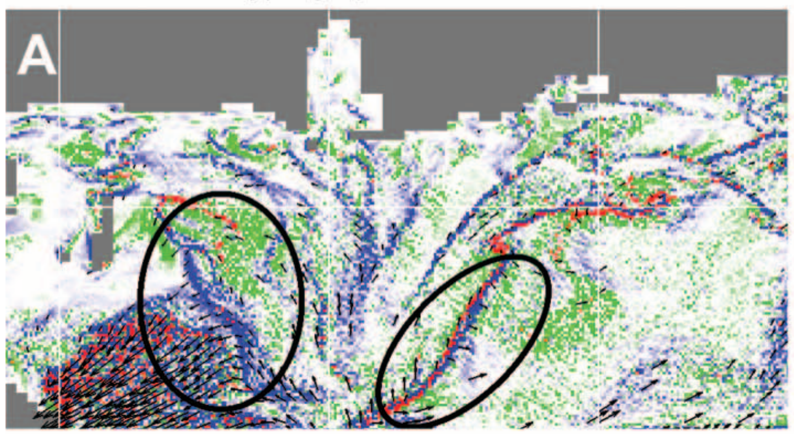
[Mezic, Loire, et al., Science, 2010]



# Mesochronic analysis of 2D flows correctly detected phenomena related to Deepwater Horizon Spill.

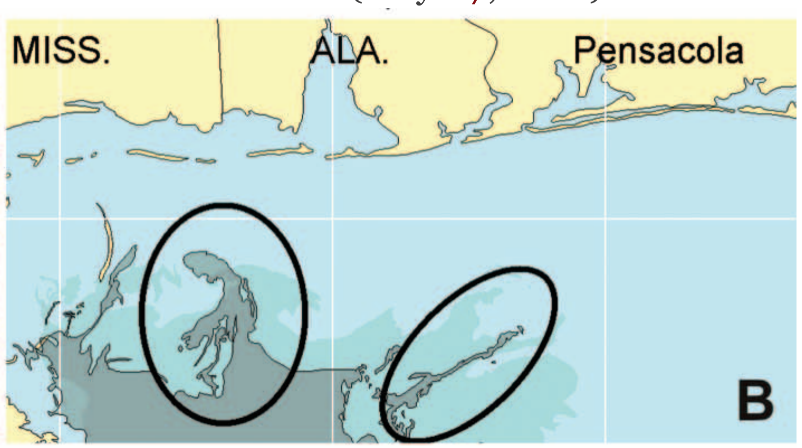
## Oil slick distribution (May, 2010):

Mesochronic Classes (May **25**, 2010)




**Prediction**

Oil Slick (May **27**, 2010)

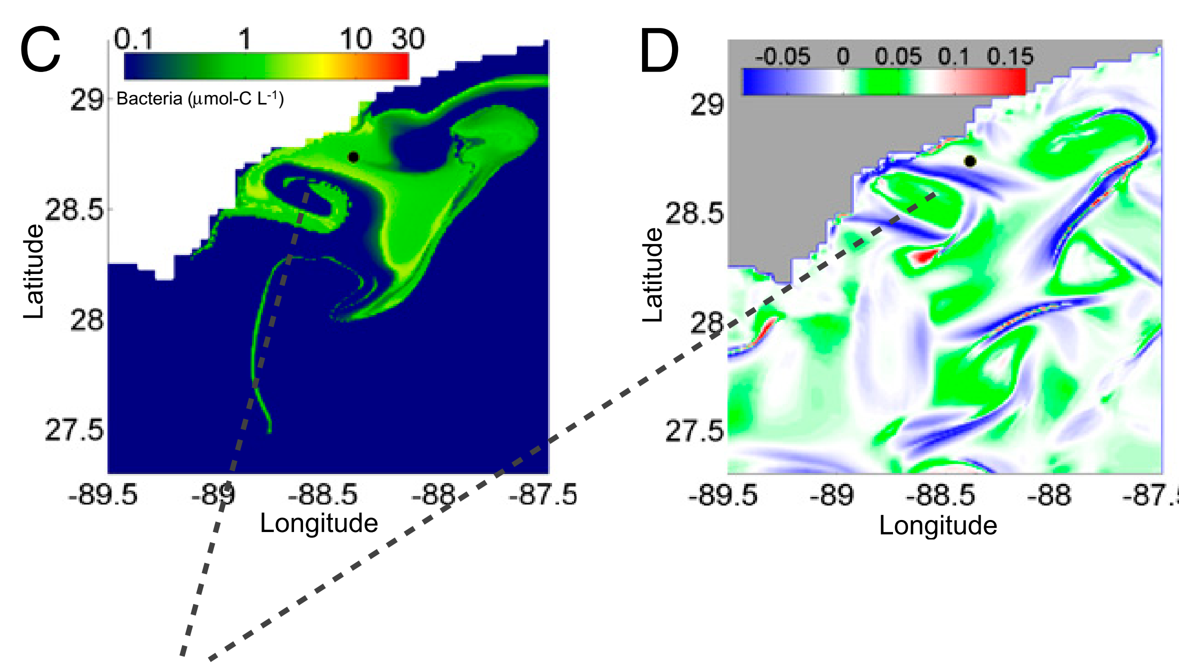



[Mezic, Loire et al., Science, 2010]

## Distribution of bacteria (Jun, 2010):

Bacterial respiration

Mesochronic Classes

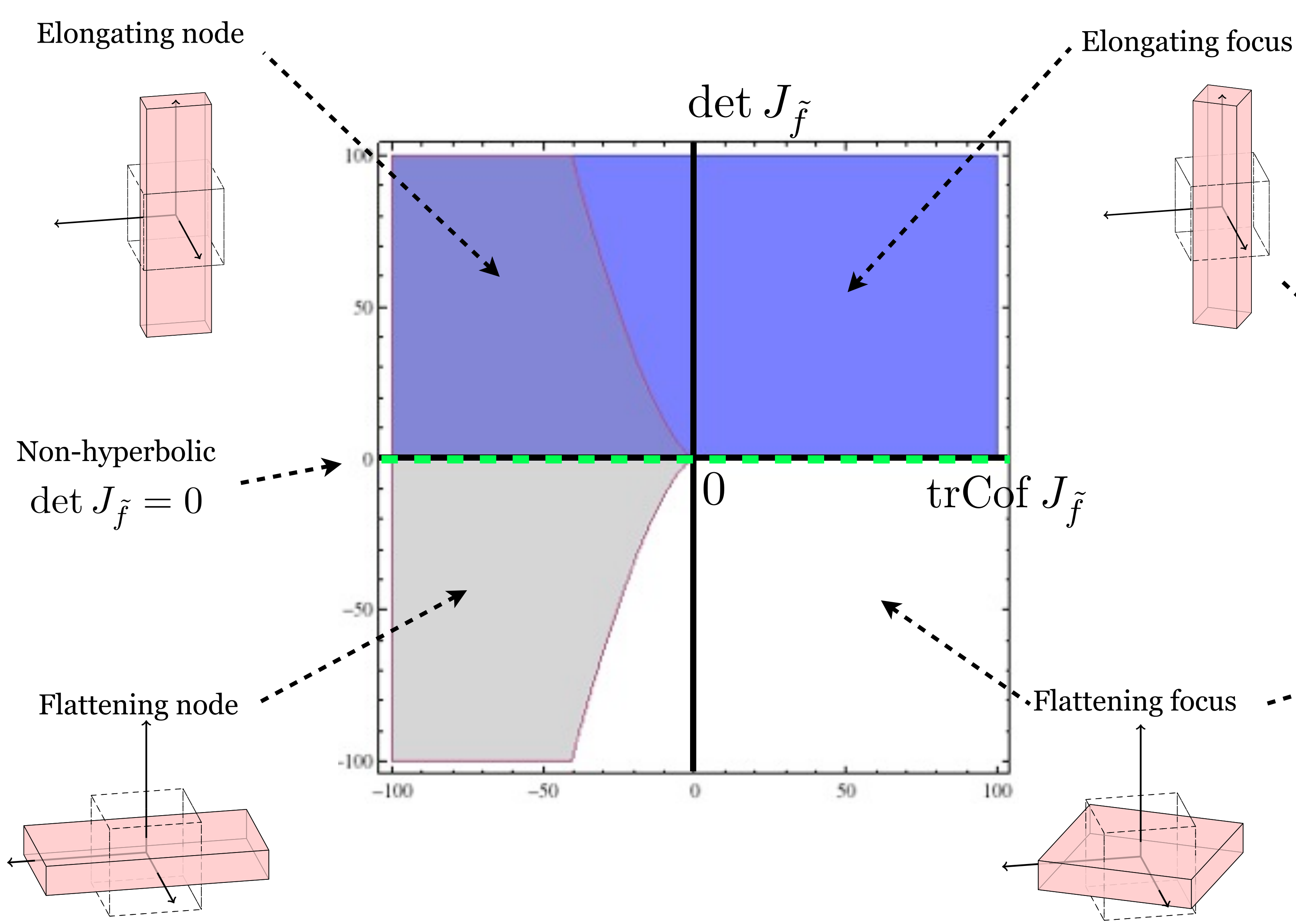



Elliptic (rotating) region.

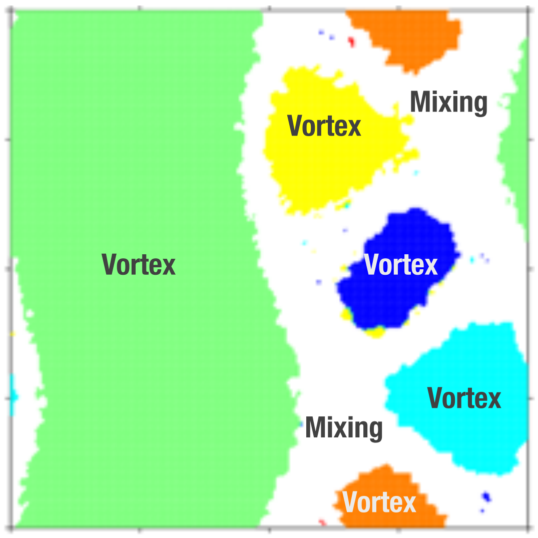
[Valentine, Mezic et al., PNAS, 2012]

# In 3D flows, behaviors are parametrized by two quantities.

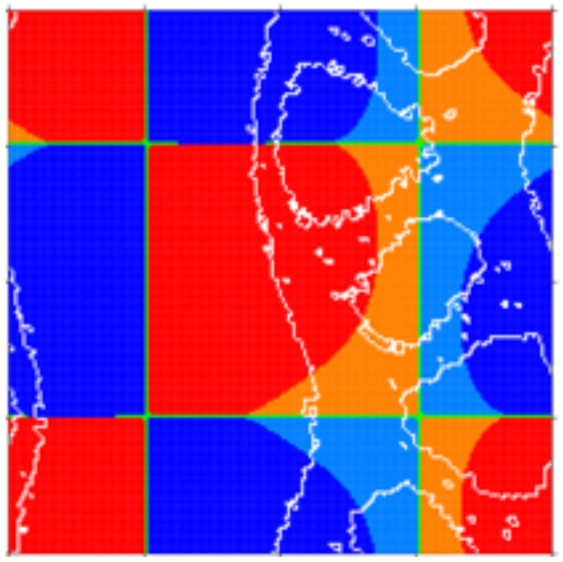
## $T = 0^+$ Okubo-Weiss-Chong:




ABC Flow ($z$=0 slice)




Okubo-Weiss-Chong



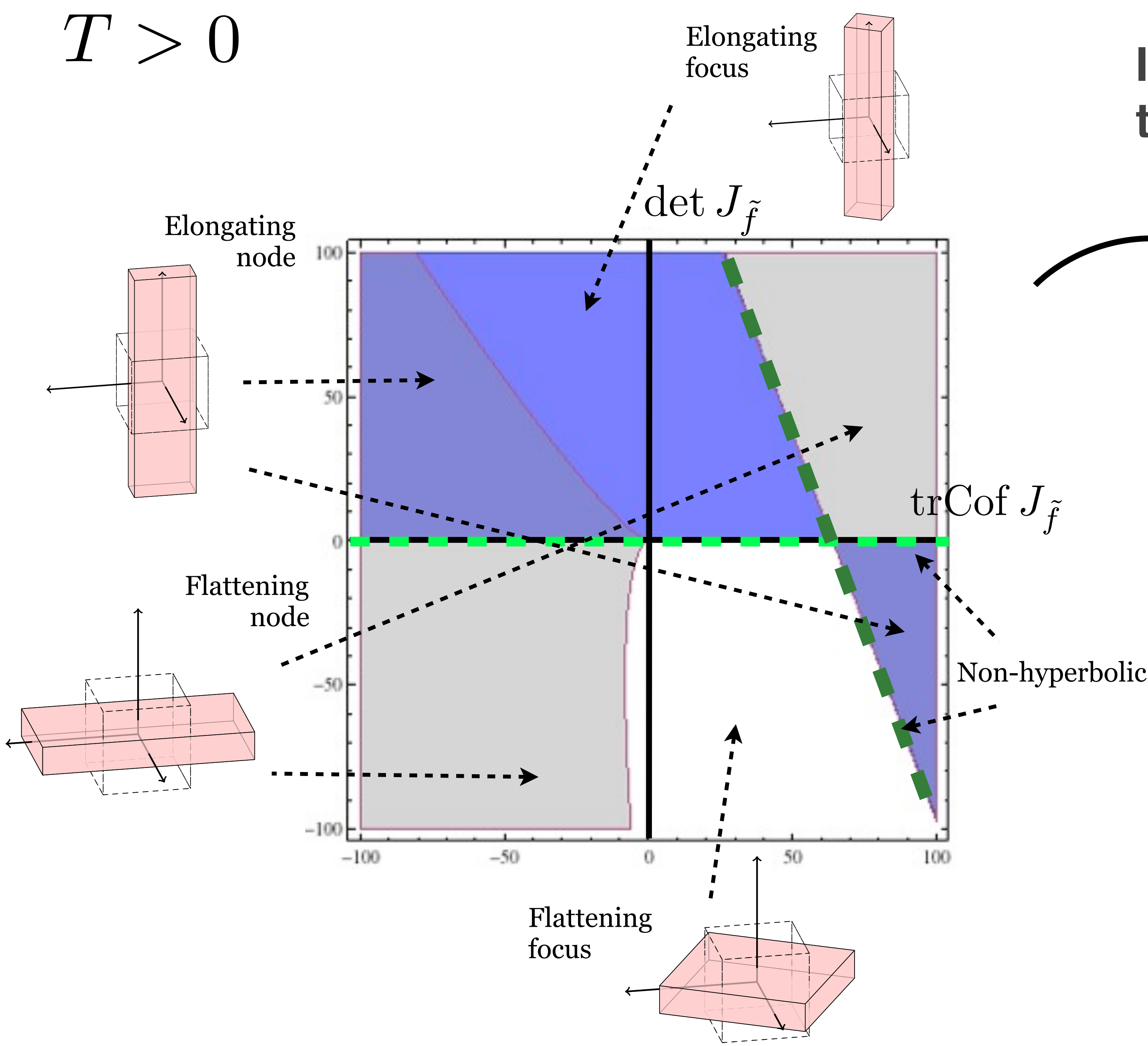
$T$=$0^+$

**Criterion yields non-intuitive results even for steady flows:**
**boundaries do not match understanding of invariant structures.**

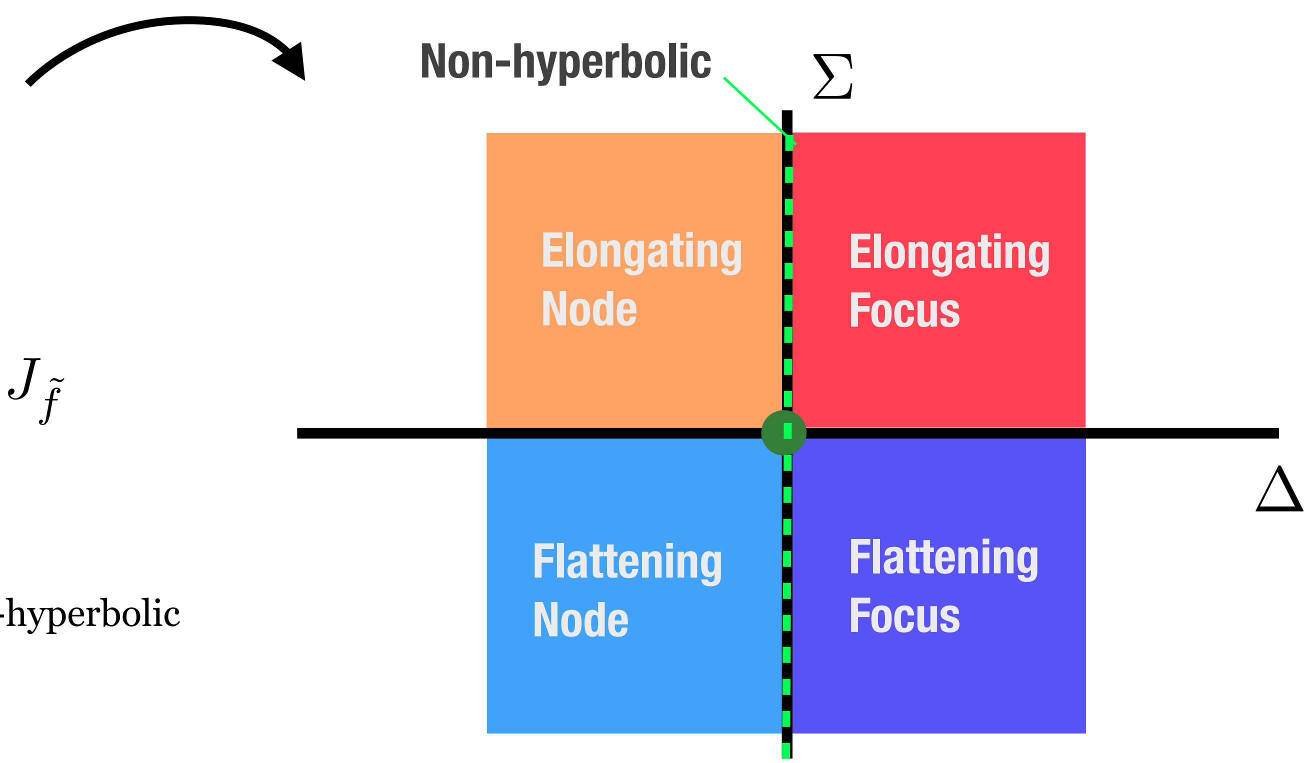
[with S. Siegmund (TU Dresden),
T.S. Doan (Imperial College London)]

# Mesochronic deformation classes can be identified by signs of two parameters.

$T > 0$




Introduce two new quantities $\Sigma\,\Delta$ that separate hyperbolic classes:

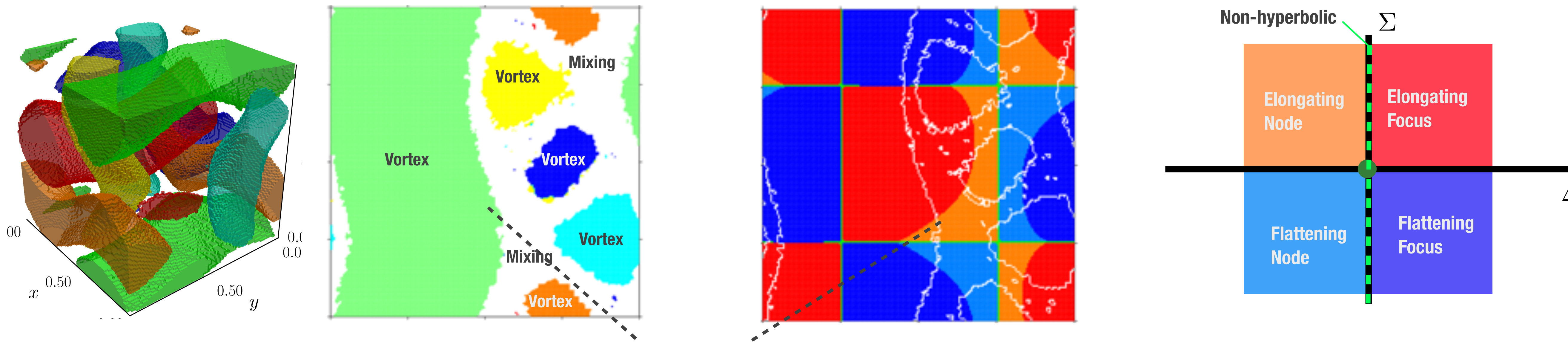



**Incompressibility:**

$$\mathrm{tr}\, J_{\tilde{f}} + T\,\mathrm{trCof}\, J_{\tilde{f}} + T^2 \det J_{\tilde{f}} = 0$$

# Analysis of ABC flow matches our intuition.

**Invariant sets ($z$=$0$ slice)** **Okubo-Weiss-Chong** **Mesochronic Classes**



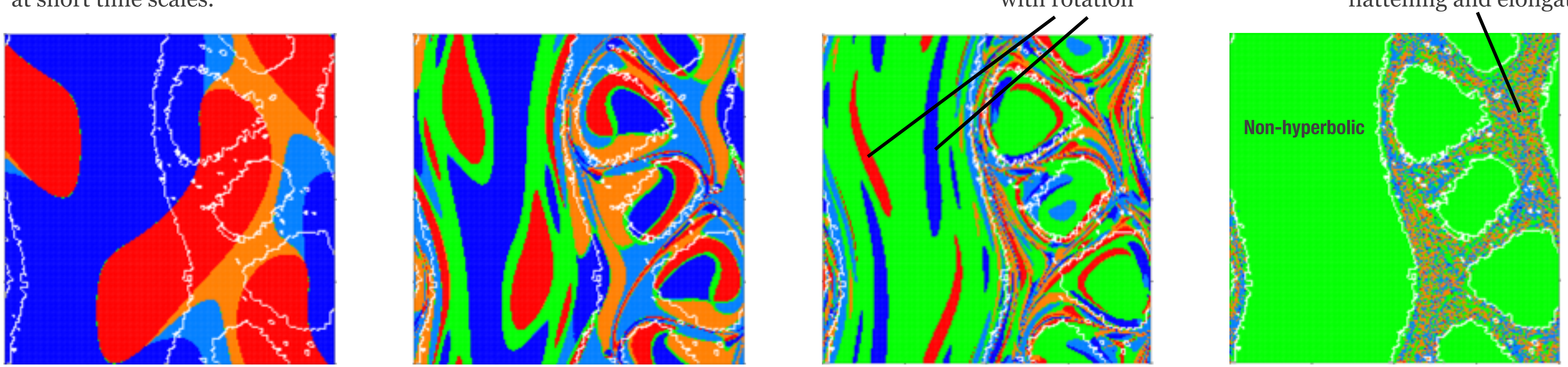

Boundaries of invariant regions

$T$=$0^+$

Hyperbolicity dominates at short time scales.

Hyperbolicity with rotation

Mixture: non-hyperbolic, flattening and elongating




$T$=$1$ $T$=$5$ $T$=$10$ $T$=$50$

# Computation is a numerical integration of evolution of the mesochronic Jacobian.

**Step 1.** Integrate a trajectory.

$$\dot{x}_p = f(t, x_p), \; x_p(0) = p, \; t \in [0, T]$$

**Step 2.** Evaluate Jacobian of the vector field along the trajectory.

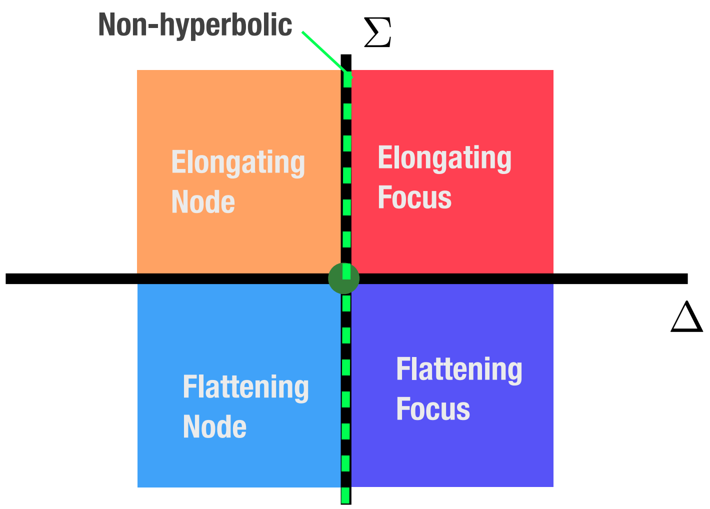
$$J(p,t) = [\nabla f]^*(t, x_p(t))$$

**Step 3.** Integrate the ODE for the mesochronic Jacobian.

Jacobian of averaged vector field
is not
average of Jacobian of vector field.

$$\frac{d}{dt} J_{\tilde{f}}(p,t) = \frac{J(p,t) - J_{\tilde{f}}(p,t)}{t} + J(p,t) J_{\tilde{f}}(p,t)$$

**Step 4.** Compute trace, determinant of mesochronic Jacobian and mesochronic classes.




$\Delta, \; \Sigma$ are rational functions of trace, determinant, and integration time.