A Koopman operator approach for identifying *2D+1+1* transport stuctures in HYCOM North Atlantic

Andrew Poje, CUNY-CSI

MURI 3D+1 Workshop - RSMAS

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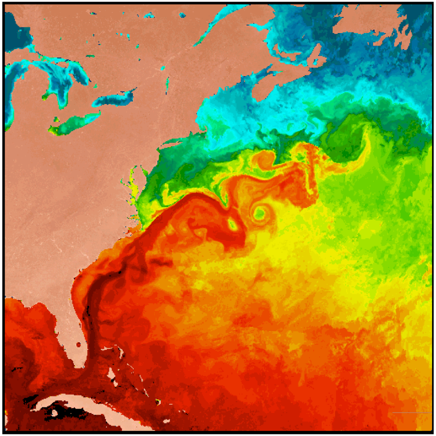
Collaborators:

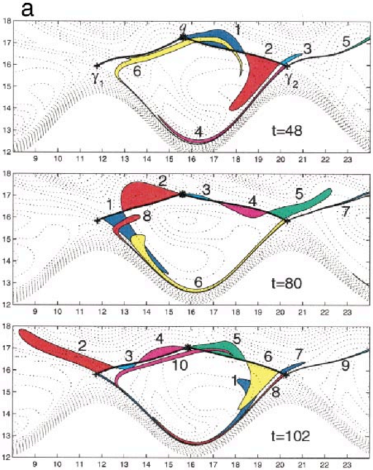
- Rakulan Sundralingam, Lucas Garber, Alexandre Fabregat – CUNY Math/Phys
- Igor Mezić, Marko Budišić – UCSB
- Angelique Haza, Tamay Özgökmen– RSMAS
- MURI 3D+1 Team

'Its deja-vu, all over again', L.P.B.

- LCS: Visualize time history of Lagrangian kinematics in Eulerian frame.
- Color/classify/delineate IC field based on common kinematics
- Define Langrangian Structures ...
- use definition to calculate transport between structures.



- Models are much better (or models have much higher resolution)
- 'Explosion of submesoscale energy' (Capet)
- Ageostrophic - $\nabla_h \mathbf{u} \sim 0$?
- Interest has shifted to smaller scales
- LCS: No longer chasing 'slow/fat', but 'fast/small'
- ... with vertical structure.



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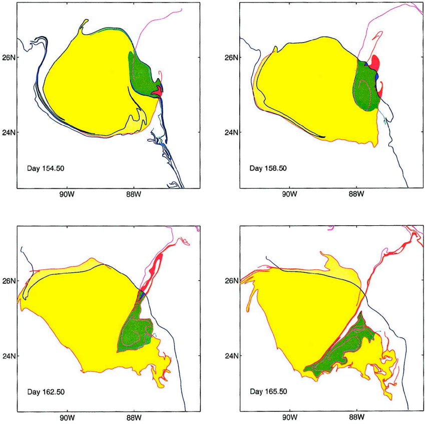

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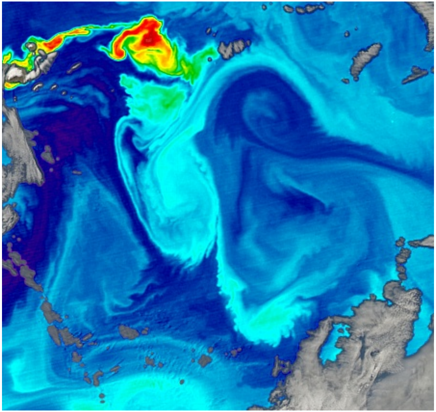

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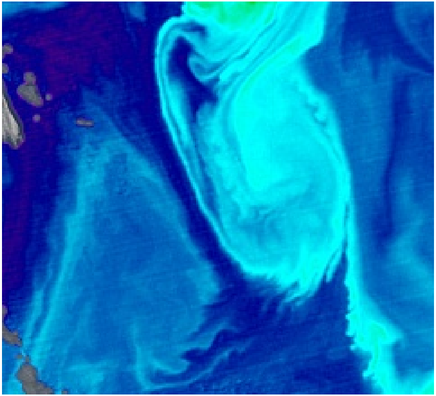

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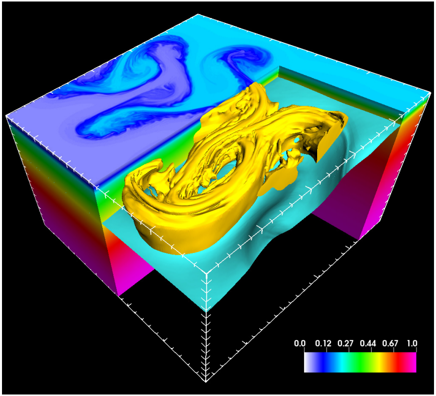
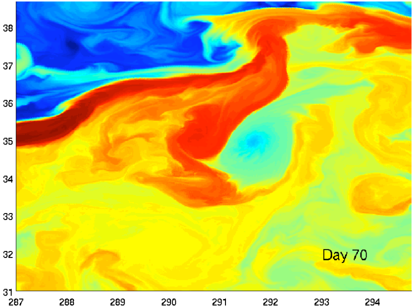


- ’Explosion’ of LCS techniques.
- Interest has shifted from: unstable (hyperbolic) structures (boundaries)
- To: stable (elliptic) structures (cores)
- Realization: ‘hyperbolic invarient manifolds are typically densely imbedded in structures’ we want.



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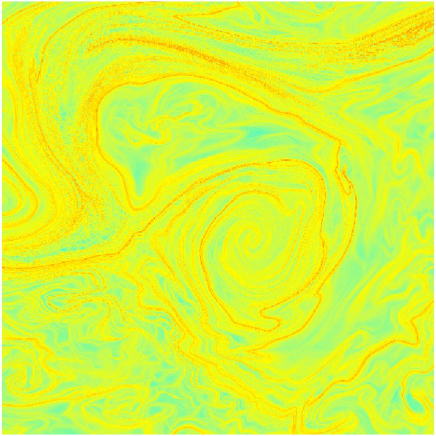
DLE: T = 8 days



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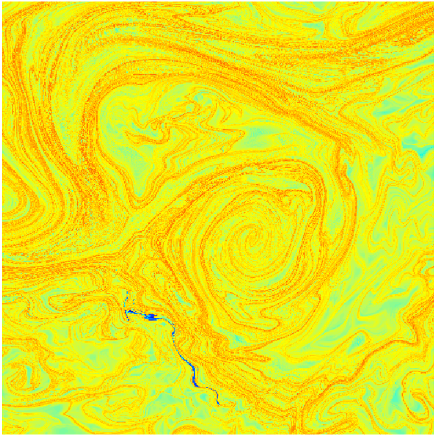
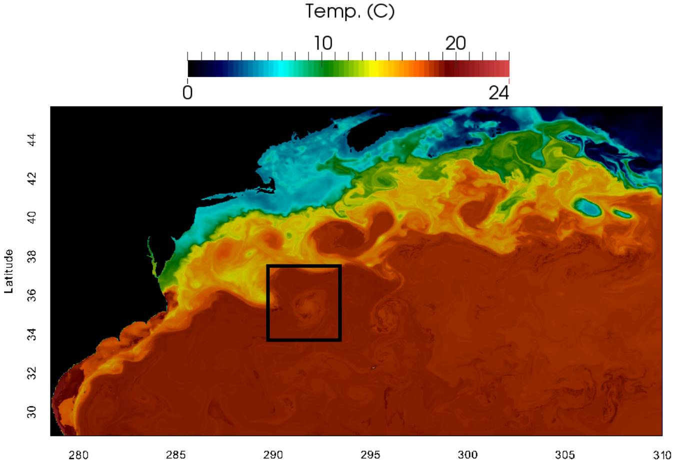
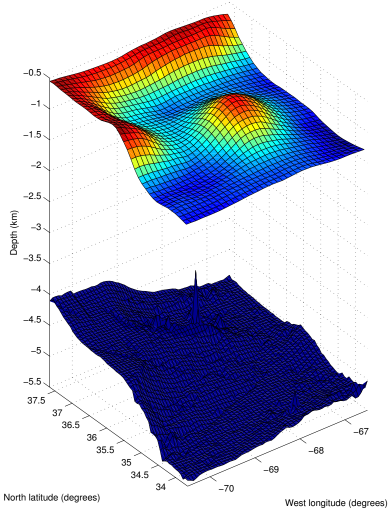


- Specifically: Dipole with strong, persistant Cold-Core Cyclone
- Eddy-jet, eddy-eddy interaction - Cross-Jet exchange
- 2D + 1 + 1: Advect on Isopycnals



- HYCOM data-set: $\Delta x \sim 1.5$ km.
- 30 isopyncal layers (surface stacked).
- Gulf stream rings with realistic size, strength, frequency.
- Energy at sub-mesoscales (< 10km) in model. (sub-mesocale eddy permitting)
- Complex, small-scale vertical tracer structures.
- Database: 12hr fields (90,000 particles)



Goal: Gulf-Stream Rings & HYCOM at 1/50 °

- HYCOM data-set: $\Delta x \sim 1.5$ km.
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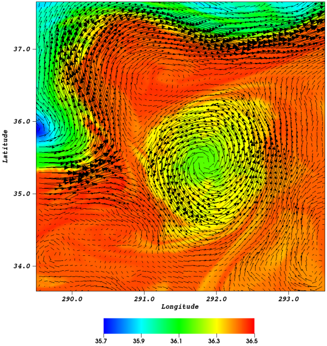
Salinity



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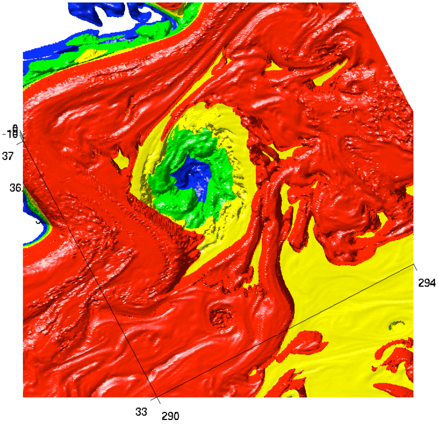
Temperature ($\Delta T = 1°$)



- Koopman Formalism: (Budišić, Mohr & Mezić)
  - Consider finite-time map (flow) $T : M \to M$
  - Koopman Operator: $U_T : \mathcal{F} \to \mathcal{F}$

$$[U_T f] = f(T(p))$$

  - Nonlinear ODES in $\mathbb{R}^n \to$ spectrum of *linear* operator U_T.
  - Koopman modes from Lagrangian averages of specified basis set …

$$f_k(x) = \mathrm{e}^{2\pi i k \cdot x}$$

  - Challenges in Ocean Context:
    - ★ Open flow: M essentially non-compact $T : M \to M'$, $M \subset M'$.
    - ★ Finite time: No well defined averaging time: $\mu(M') = g(t)$
    - ★ Open flow and/or Finite time $\to$ well defined partition? No strictly *invarient sets*.

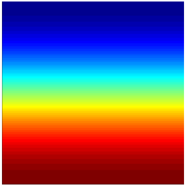
Lagrangian kinematics via time-averaged *Eulerian Observables*

- Chose set of *observables*: $f_k(\mathbf{x})$
- Examine:

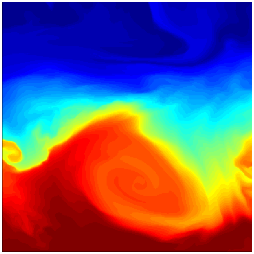
$$\overline{f_k(\mathbf{x}_0)}_\tau = \frac{1}{\tau}\int_0^\tau f_k(T(\mathbf{x}_0))dt$$

- For finite-time, non-time-periodic flows, chose
  - Time-scale via τ.
  - (Nominal) length-scale via $f_k(\mathbf{x})$.

$f_1 = -\sin(y/2)$



$\overline{f_1}, \tau = 04$ days



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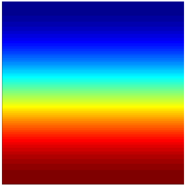
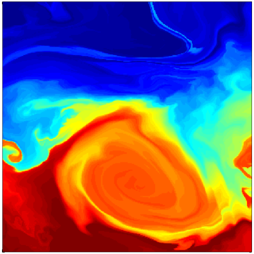


τ = 08 days



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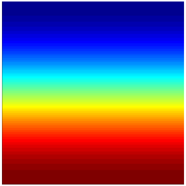

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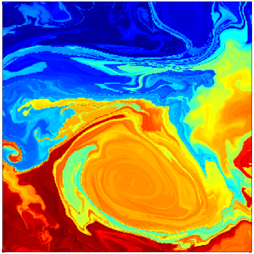

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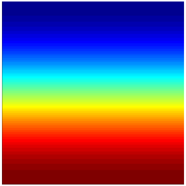
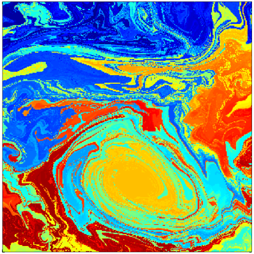


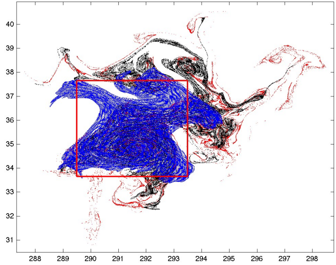
τ = 32 days



- Issue: Disparate average **v** with depth.
- Disparate average trajectory length.
- Fix: Depth dependent averaging times.

$$\tau = \tau(\Delta\rho)$$

T = 7 days: 3 Layers



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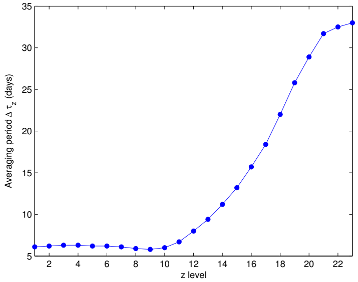
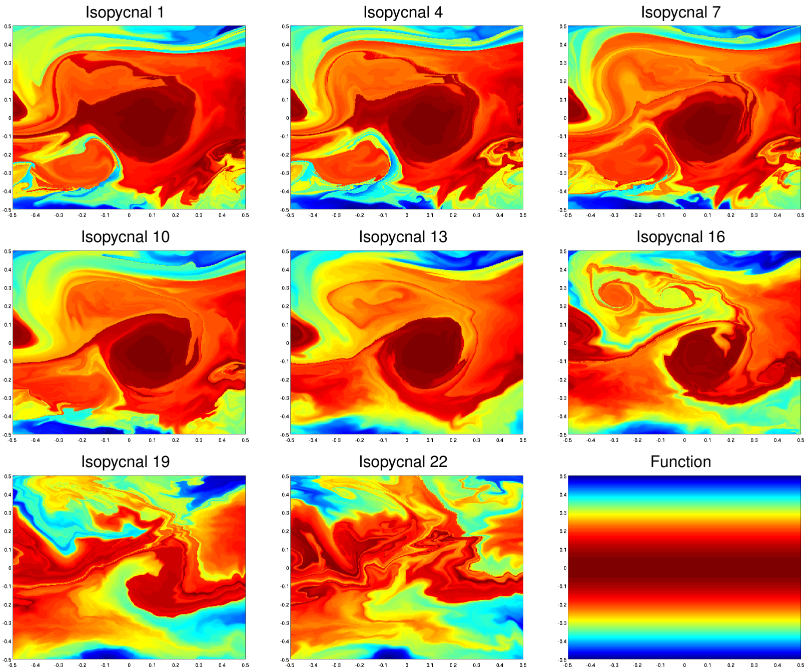
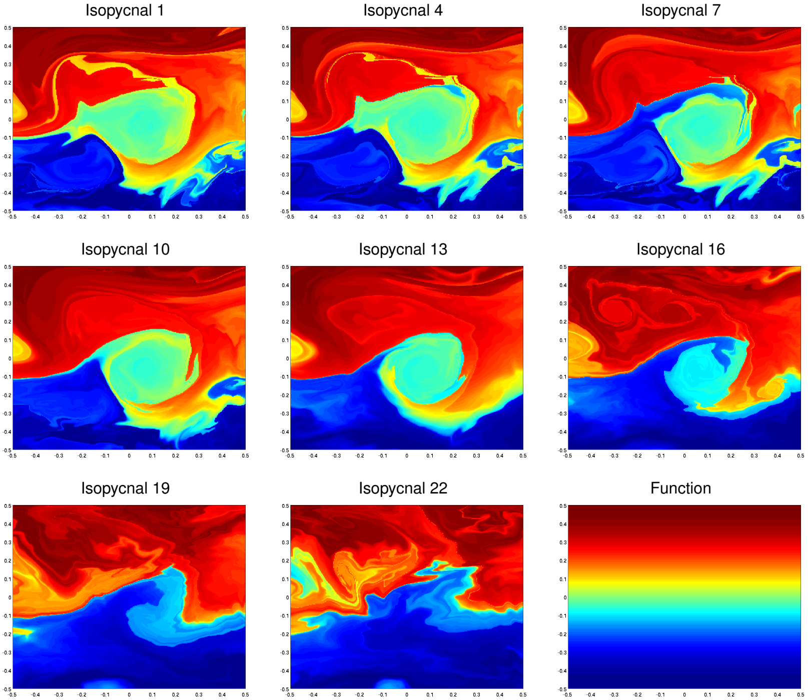



Isopycnal 1
Isopycnal 4
Isopycnal 7
Isopycnal 10
Isopycnal 13
Isopycnal 16
Isopycnal 19
Isopycnal 22
Function


Isopycnal 1
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Isopycnal 13
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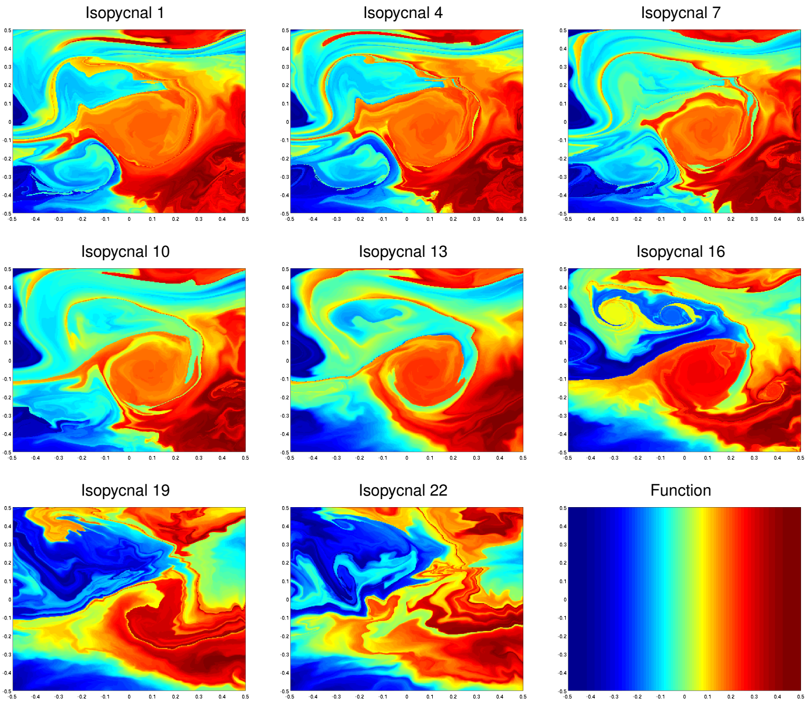
Function
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5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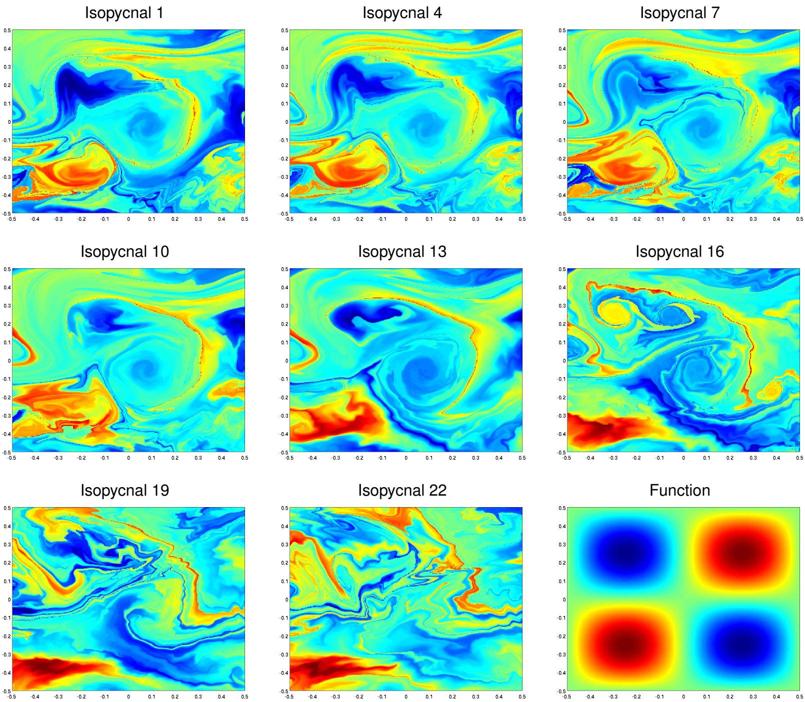
Function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Isopycnal 13
Isopycnal 16
Isopycnal 19
Isopycnal 22
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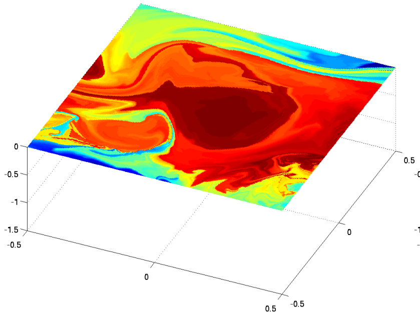
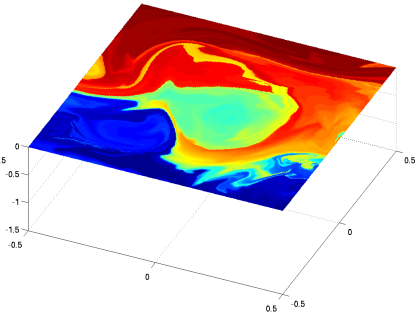
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5

Depth dependence: Observable $k_x = 0$, $k_y = 1/2$





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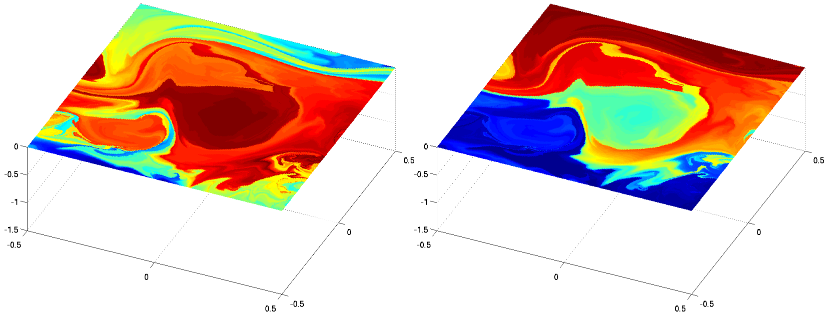

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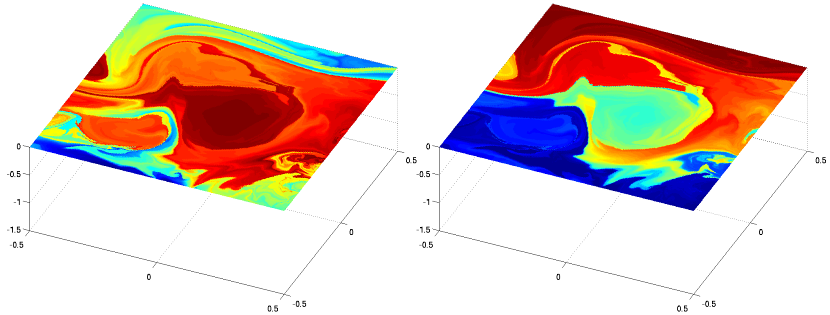

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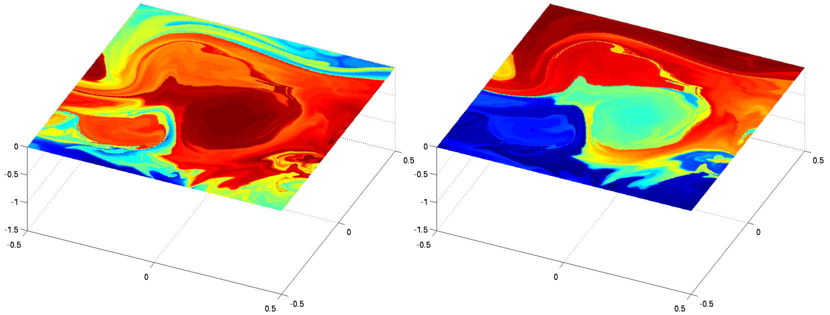

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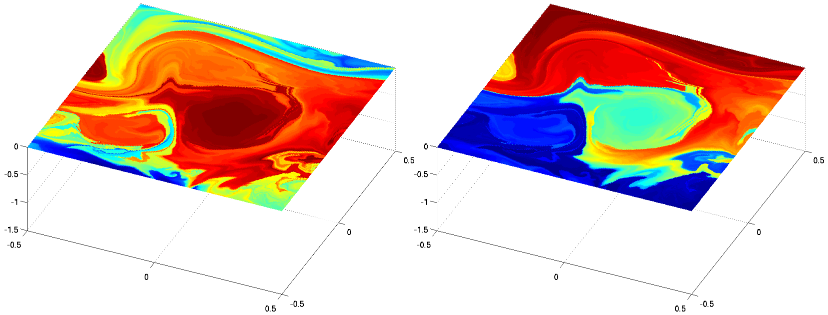

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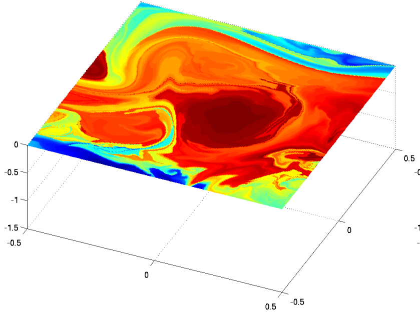

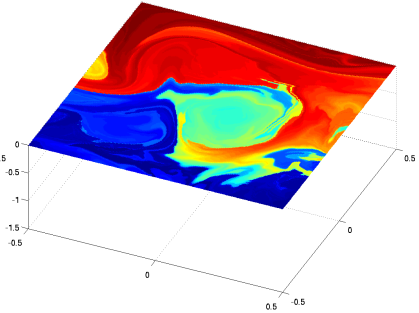

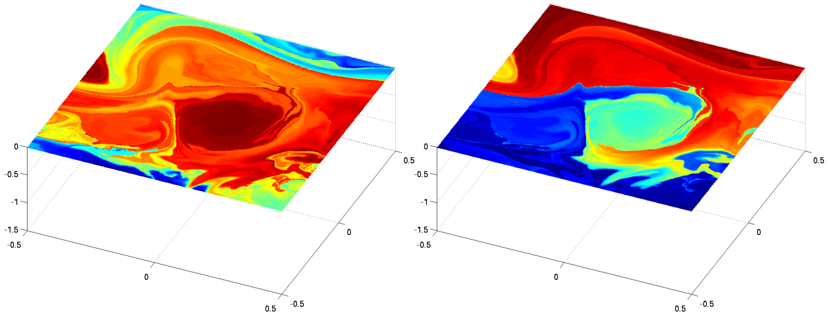
0
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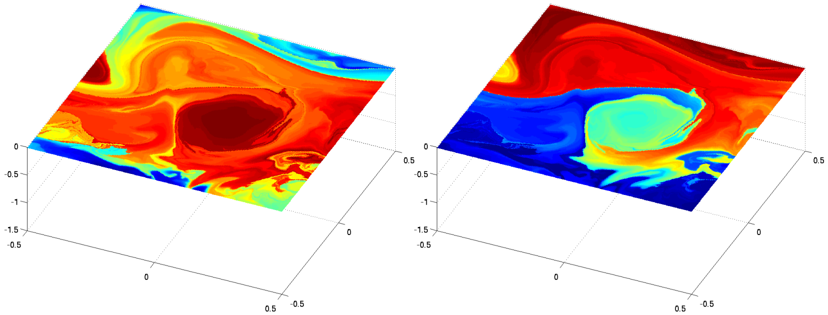

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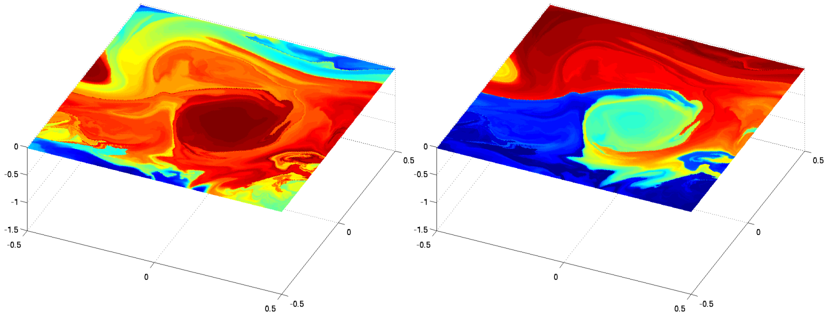

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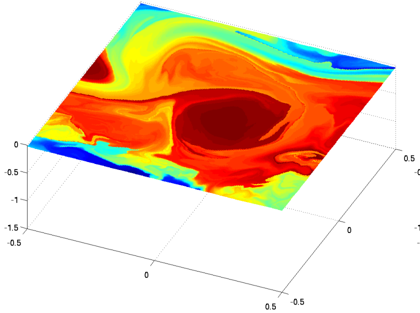

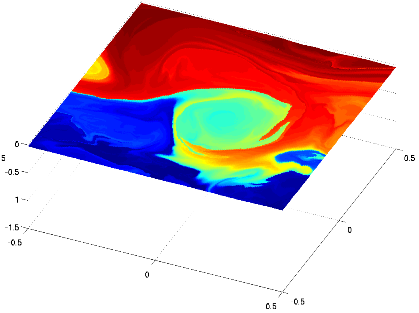

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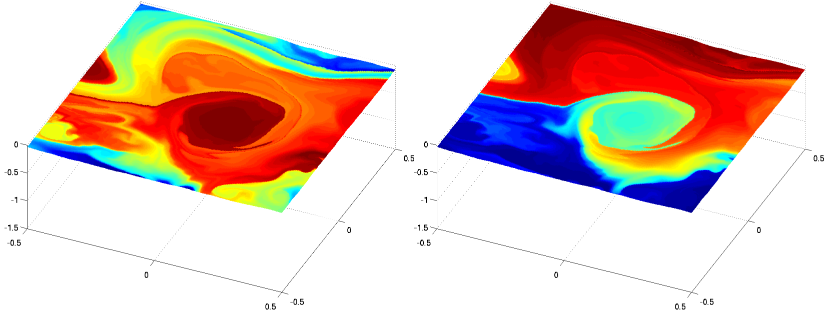

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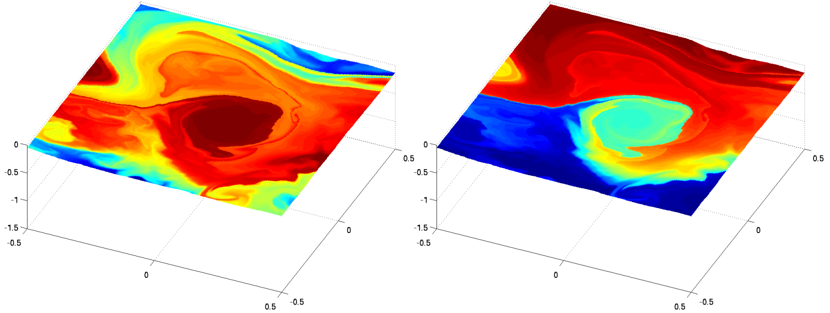

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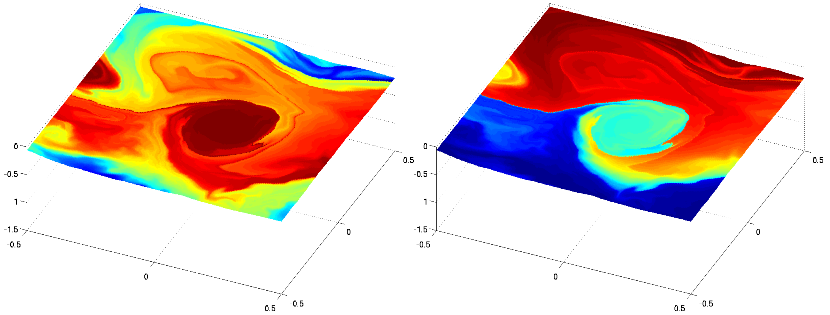

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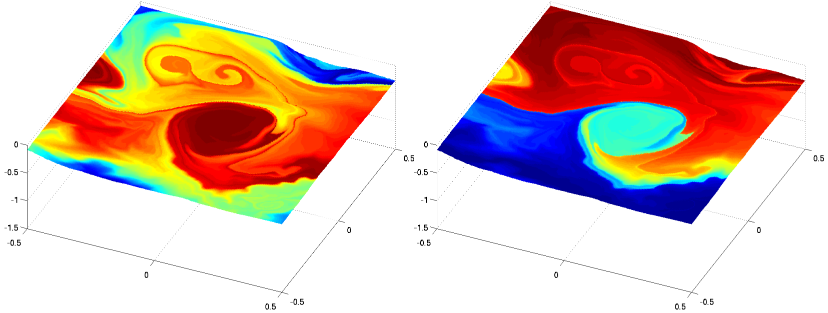

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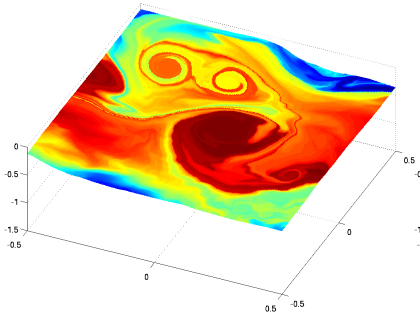

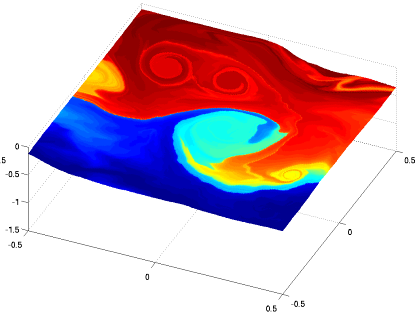

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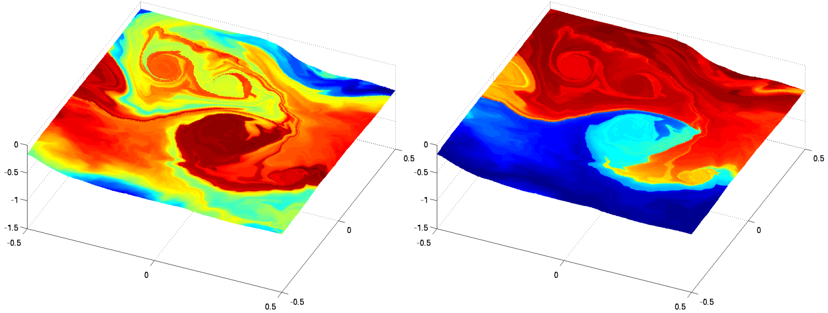

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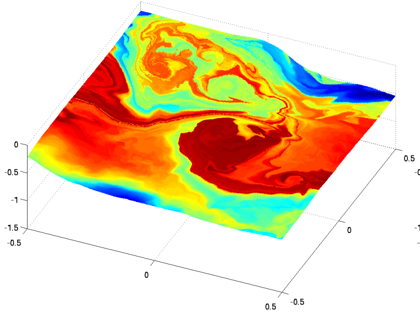
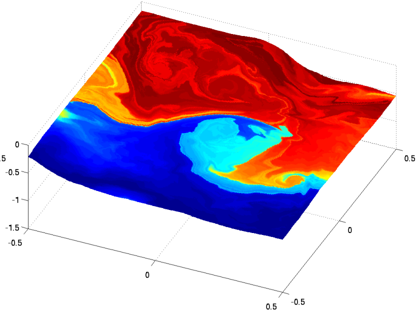
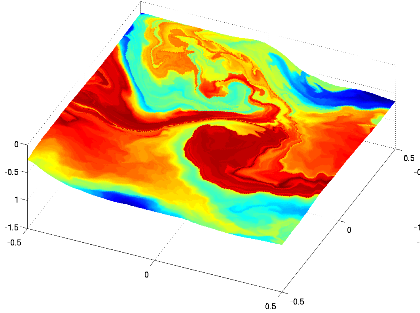
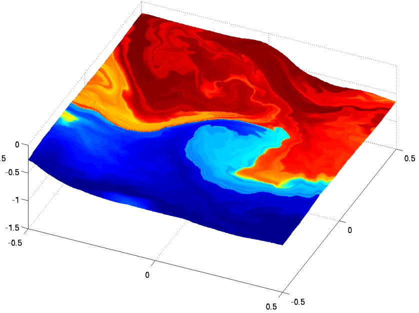


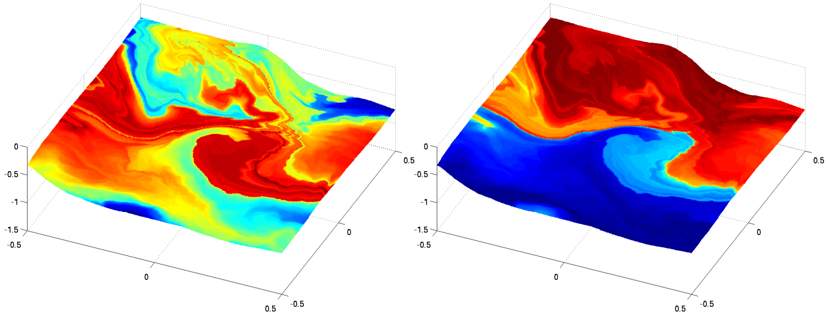


Depth dependence: Observable $k_x = 0$, $k_y = 1/2$

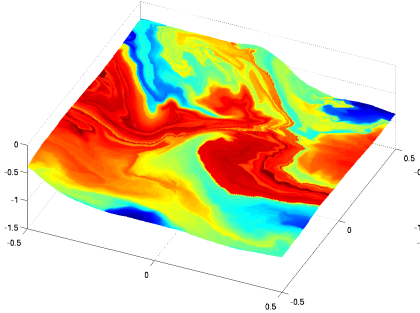


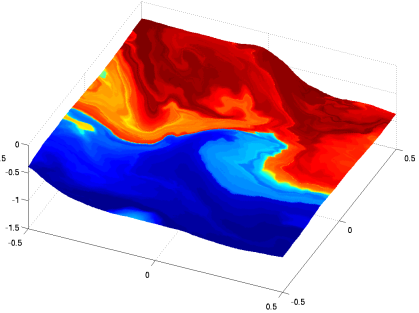


Depth dependence: Observable $k_x = 0$, $k_y = 1/2$

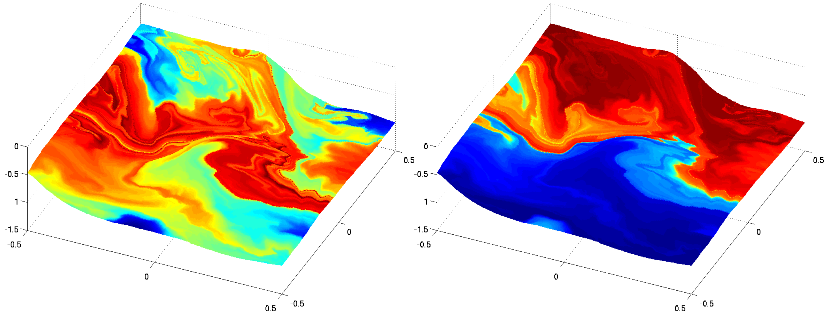


Depth dependence: Observable $k_x = 0$, $k_y = 1/2$

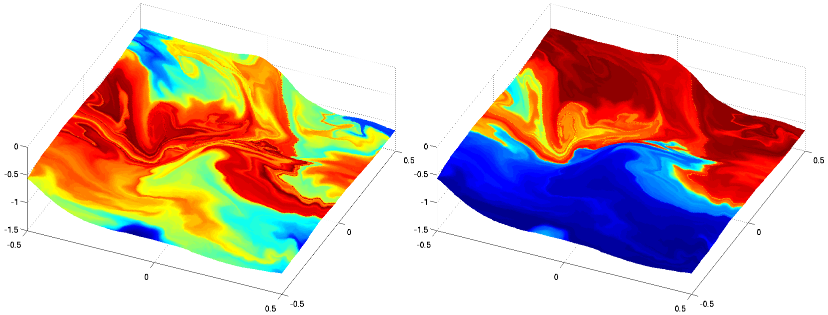




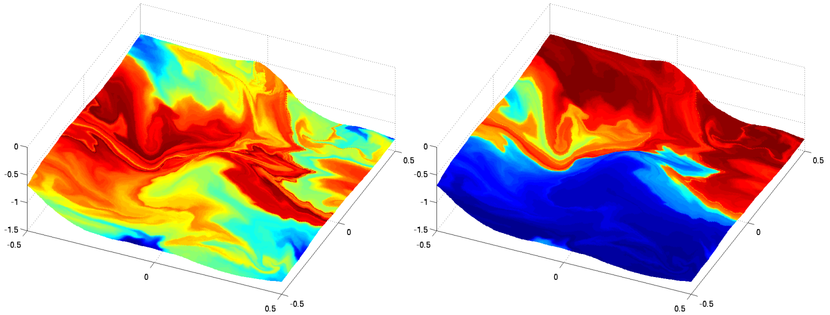
Depth dependence: Observable $k_x = 0$, $k_y = 1/2$



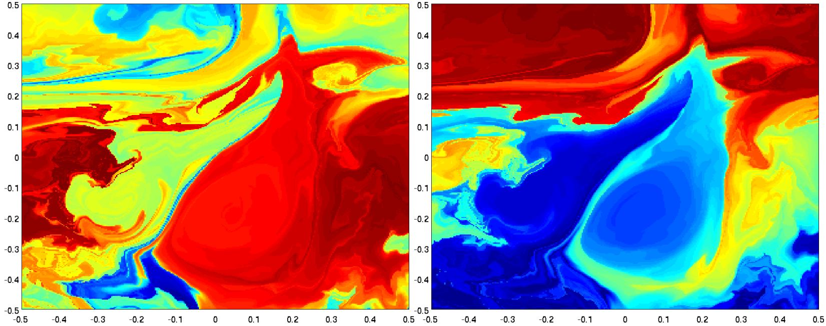
Depth dependence: Observable $k_x = 0$, $k_y = 1/2$



Depth dependence: Observable $k_x = 0$, $k_y = 1/2$



DAY=70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0.3
0.2
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-0.2
-0.3
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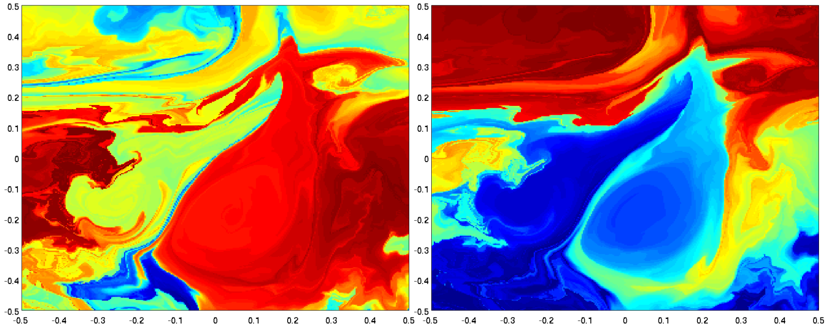
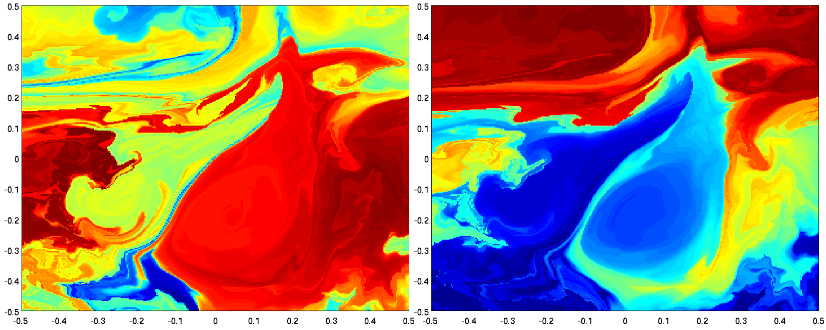
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5

DAY=70



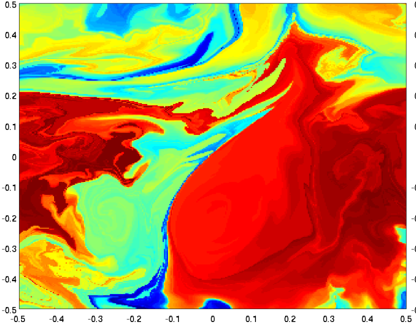
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

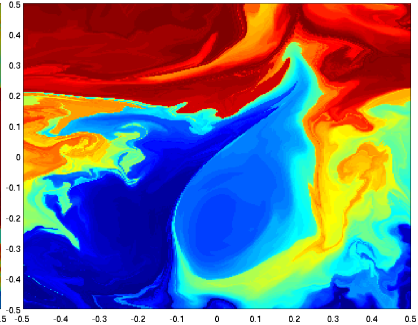
DAY=70

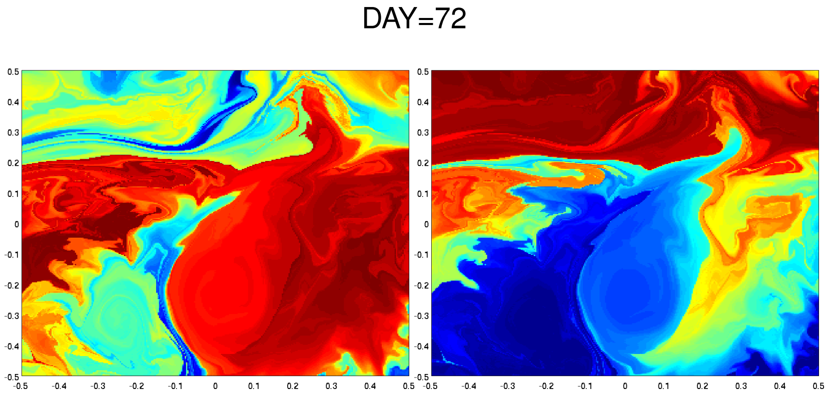


Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=71






DAY=72
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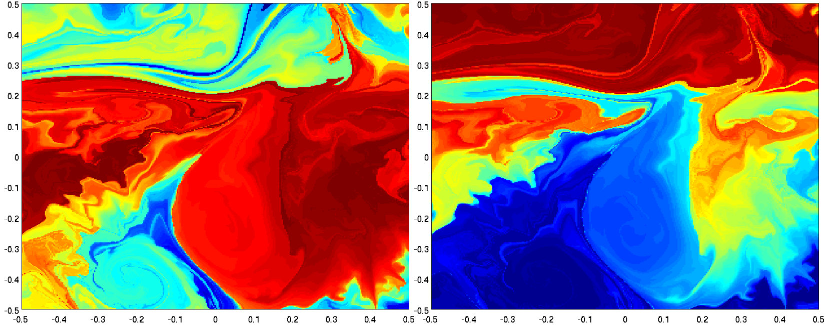
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5

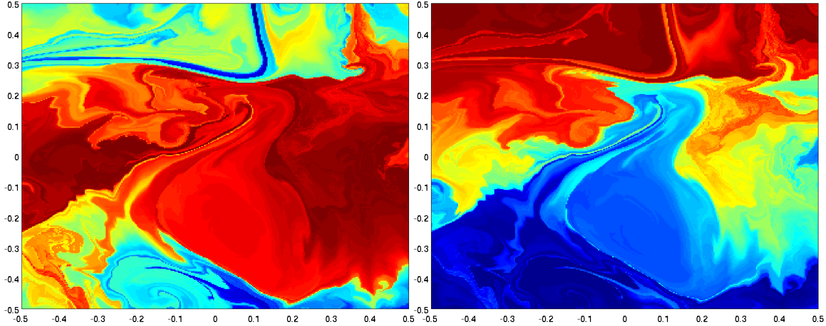
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=73



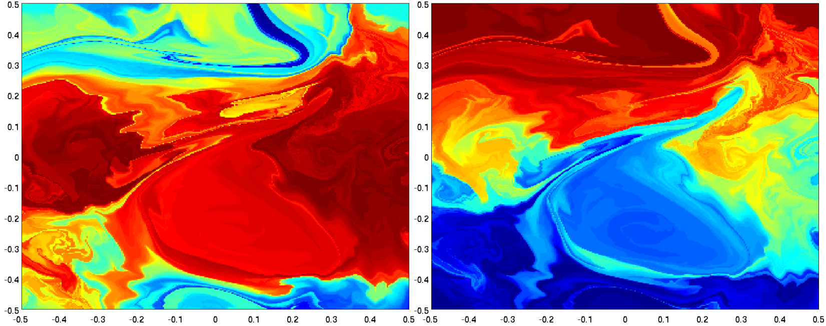
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=74



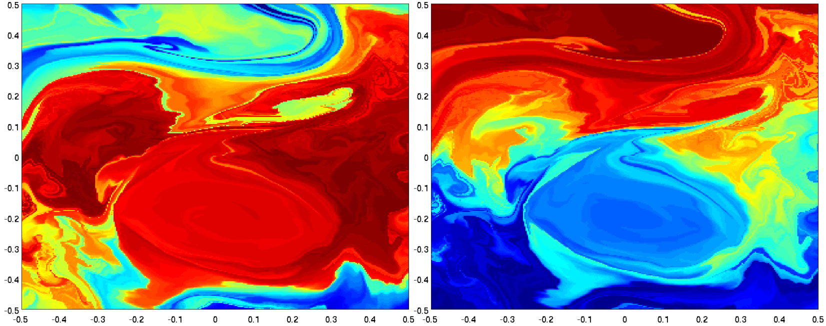
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=75



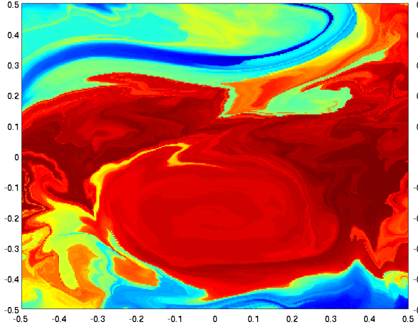
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

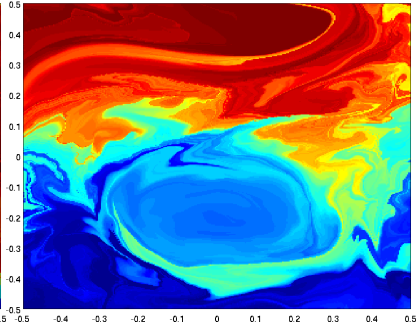
DAY=76



Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

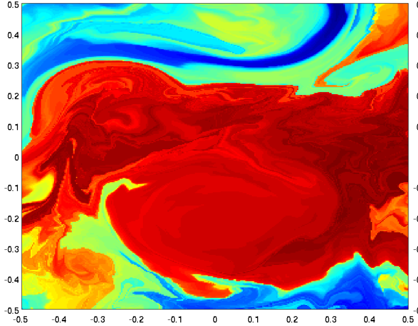
DAY=77

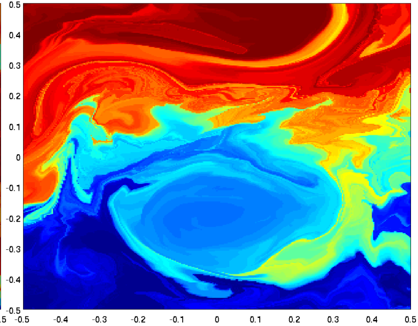




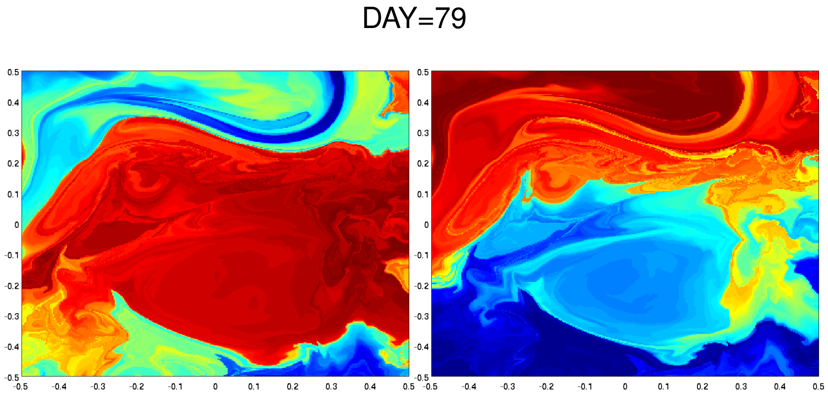
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=78



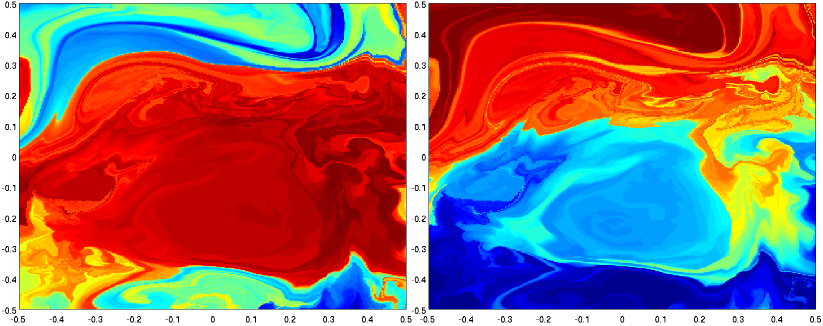


Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

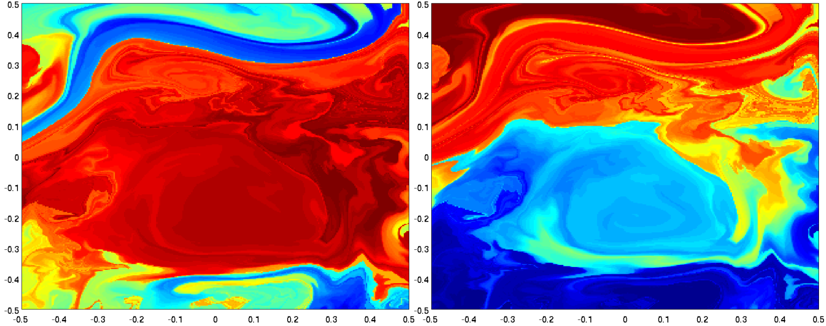


Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=80

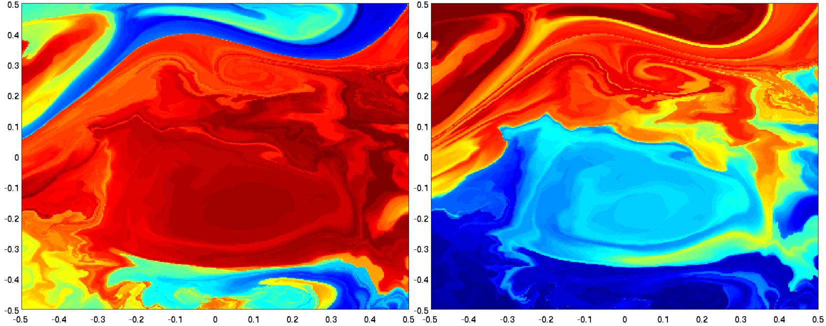


DAY=81

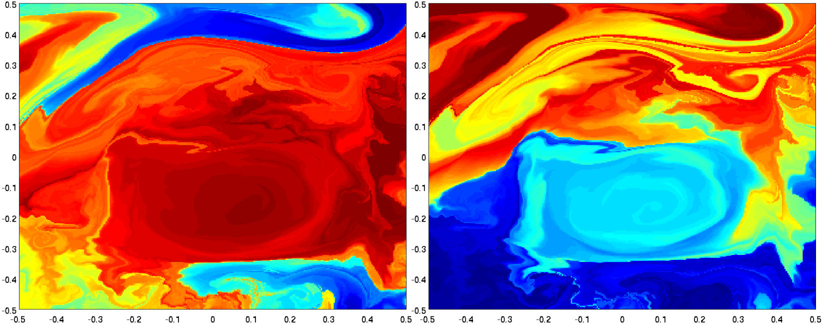


Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

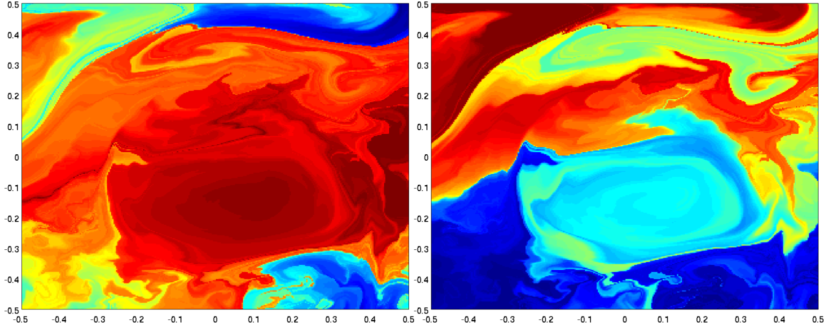
DAY=82



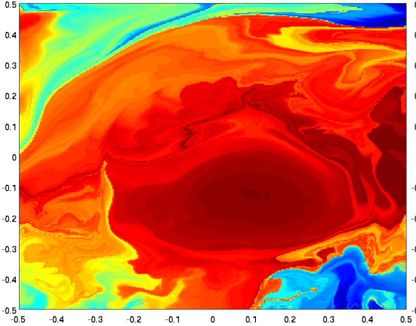
DAY=83



DAY=84



DAY=85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0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
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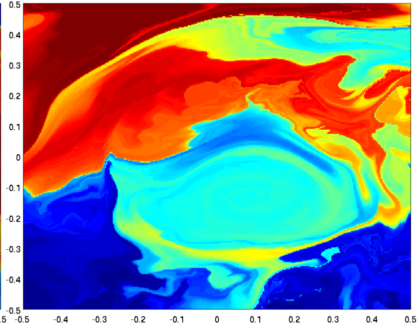
0.5
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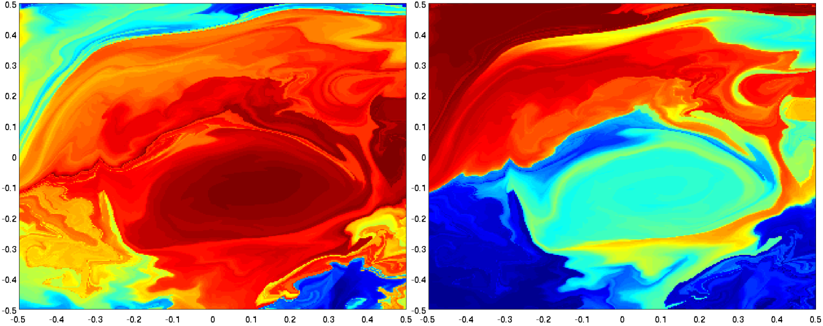
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5

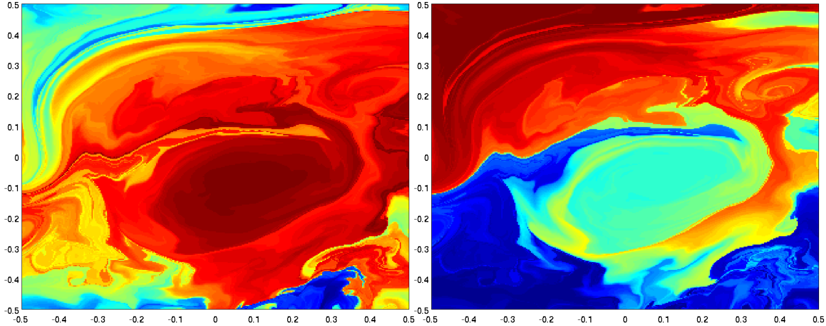
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=86



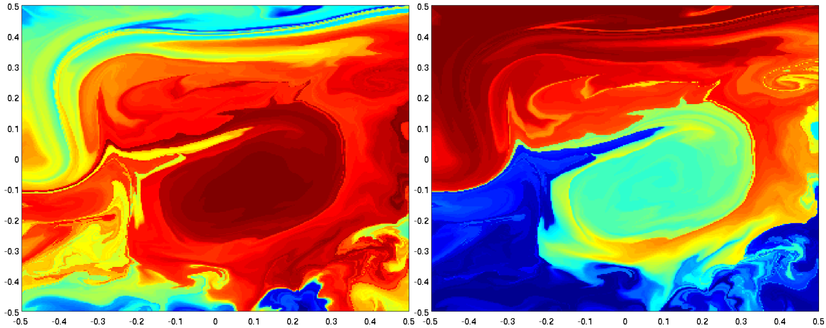
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=87

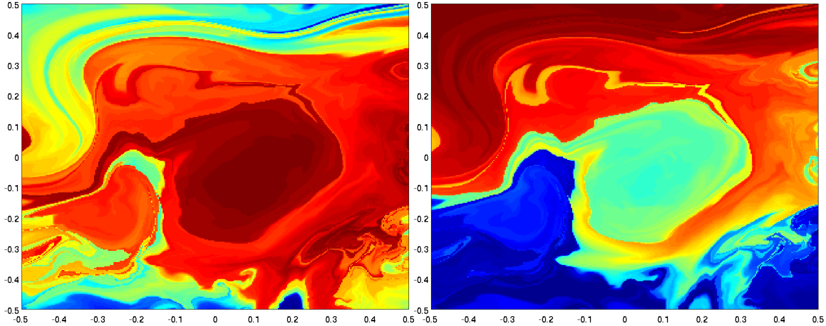


Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

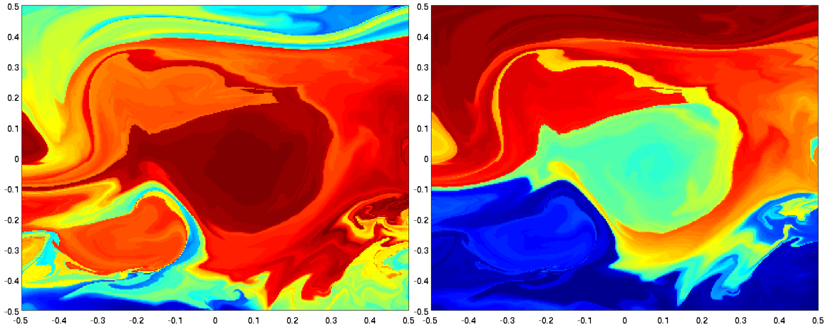
DAY=88



DAY=89



DAY=90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0.3
0.2
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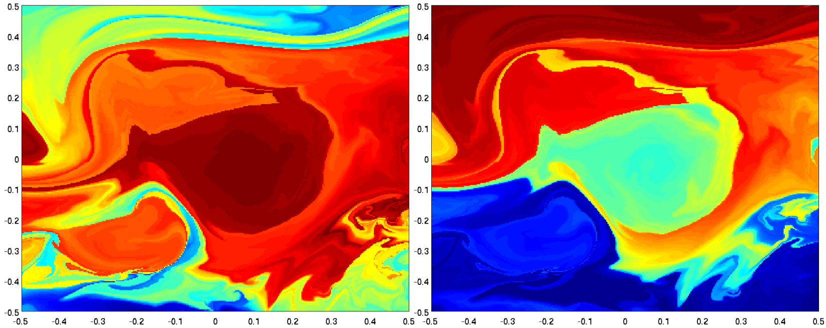
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5

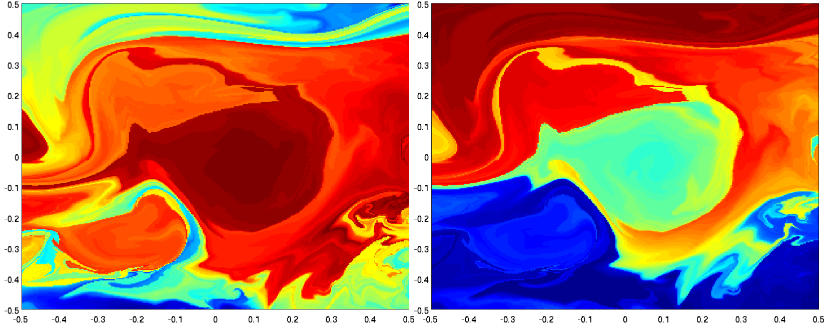
Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

DAY=90

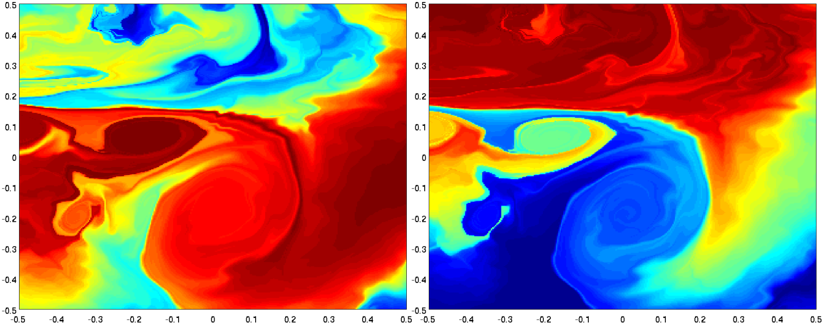


Time Dependence: Near Surface: $k_x = 0$, $k_y = 1/2$, Isopycnal 2

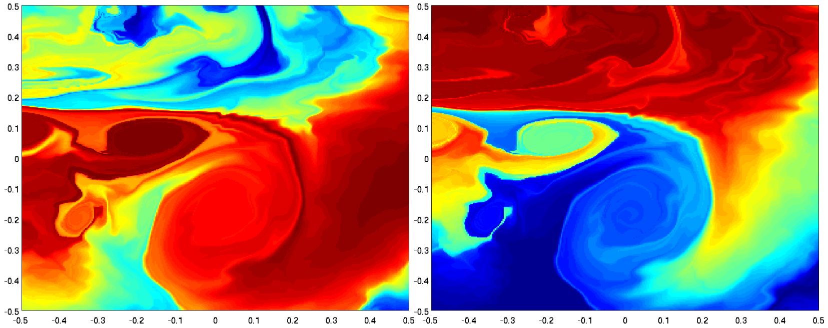
DAY=90



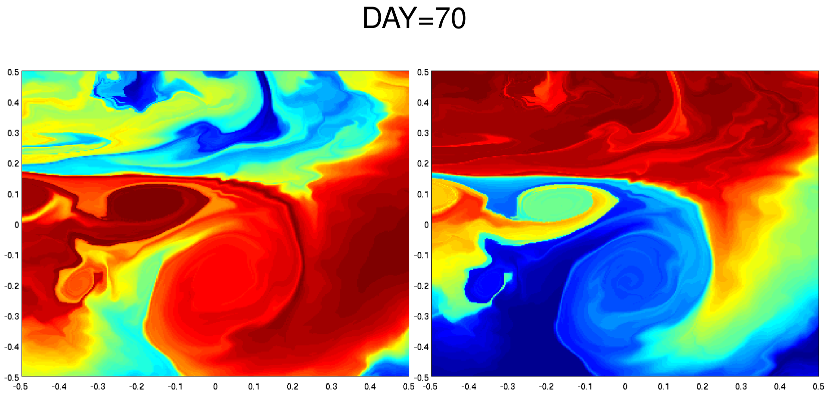
DAY=70



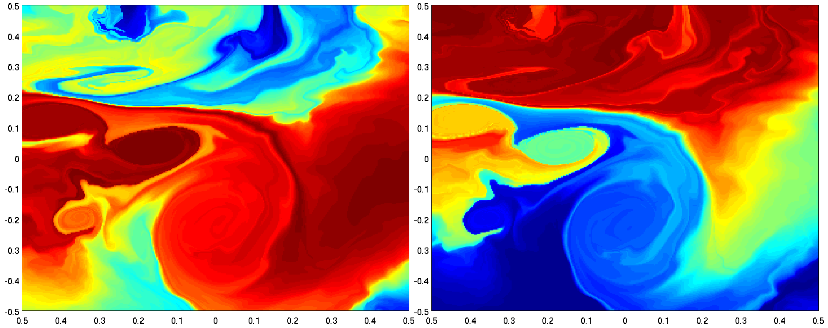
DAY=70



Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

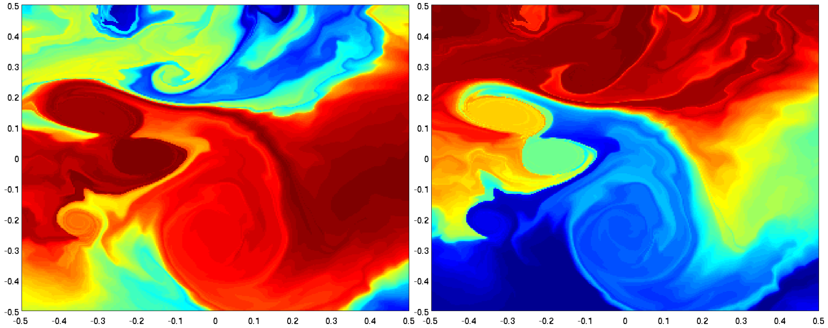


DAY=71

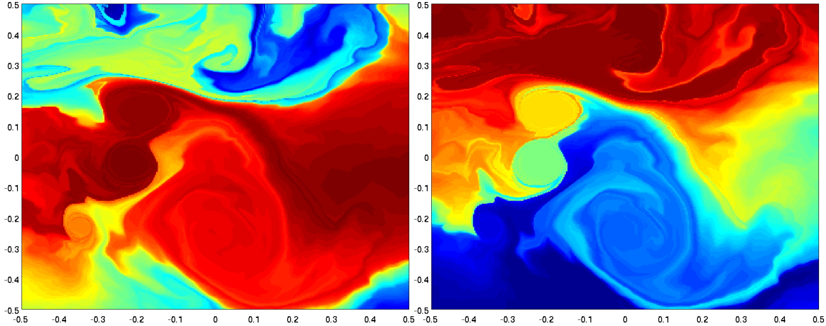


Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=72

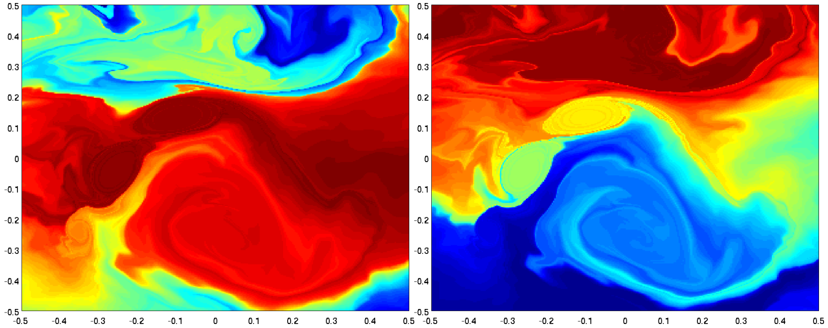


DAY=73



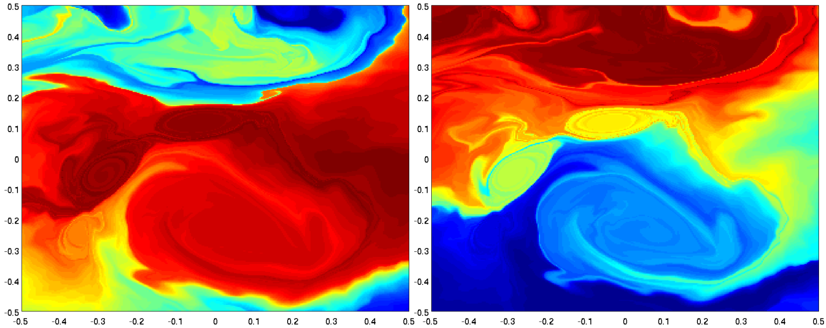
Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=74

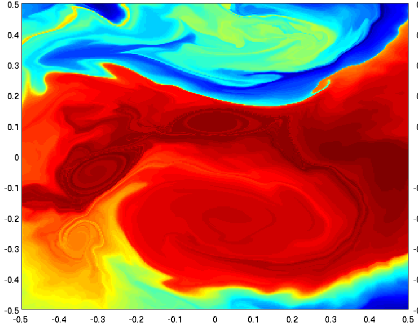


Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=75



DAY=76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0.3
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-0.1
-0.2
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-0.1
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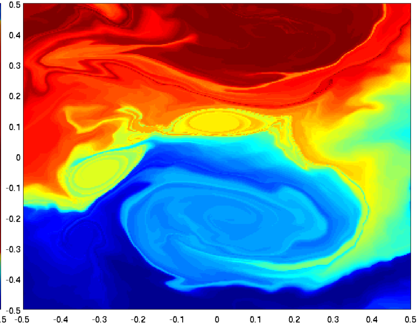
0.1
0.2
0.3
0.4
0.5


0.5
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0.2
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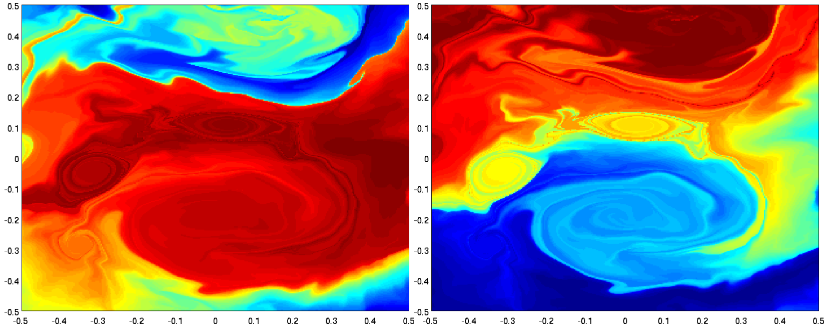
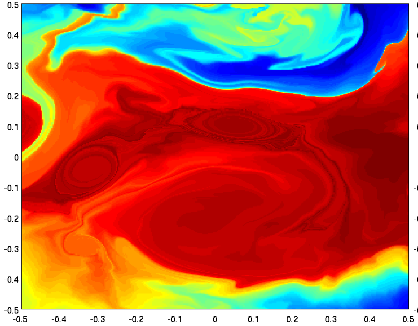
0.4
0.5

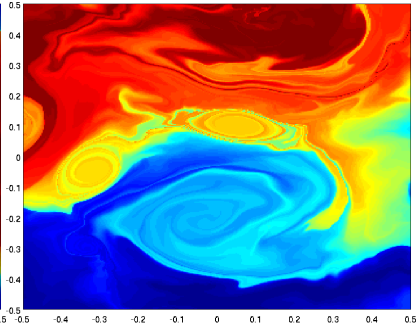
DAY=77



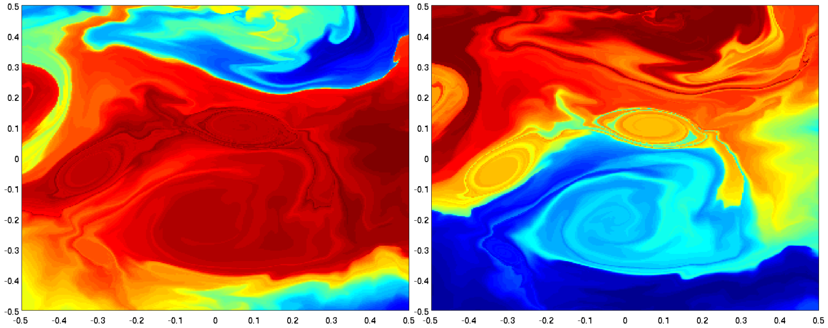
Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=78





DAY=79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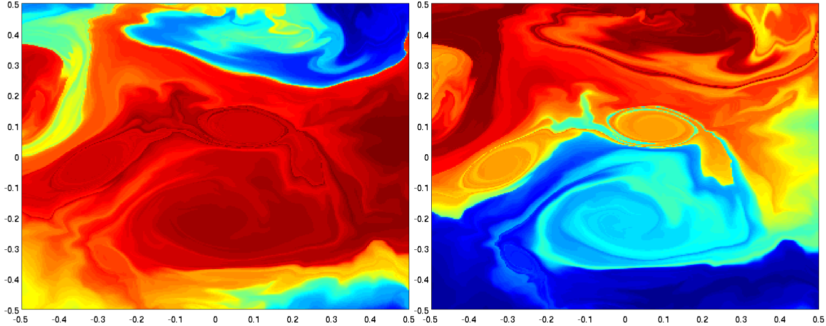
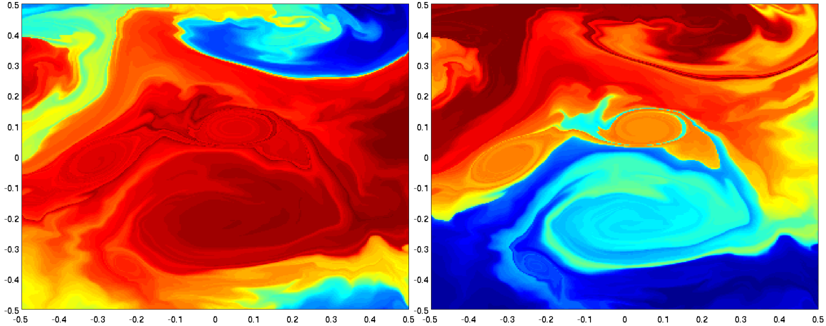
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5
-0.5 -0.4 -0.3 -0.2 -0.1 0 0.1 0.2 0.3 0.4 0.5

Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=80



DAY=81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-0.2
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0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
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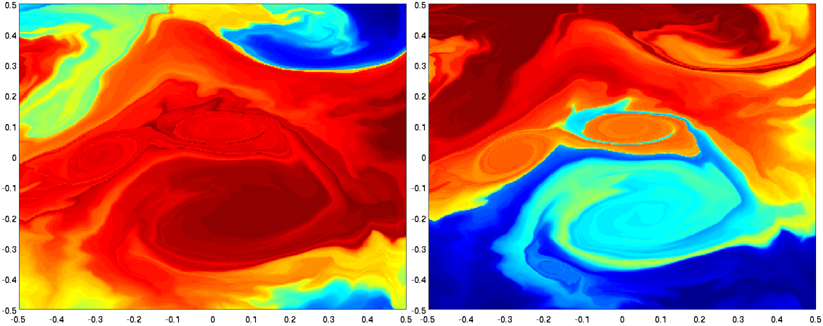
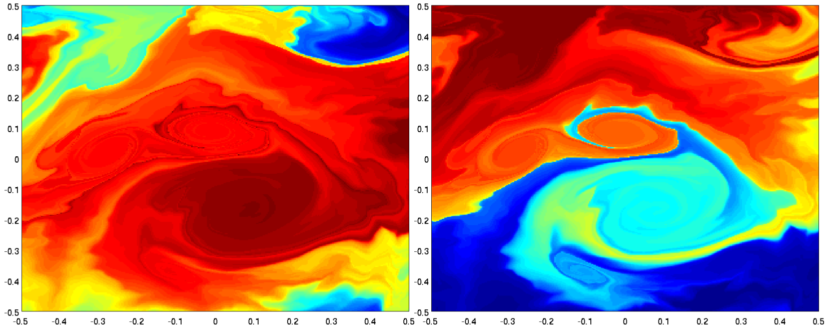
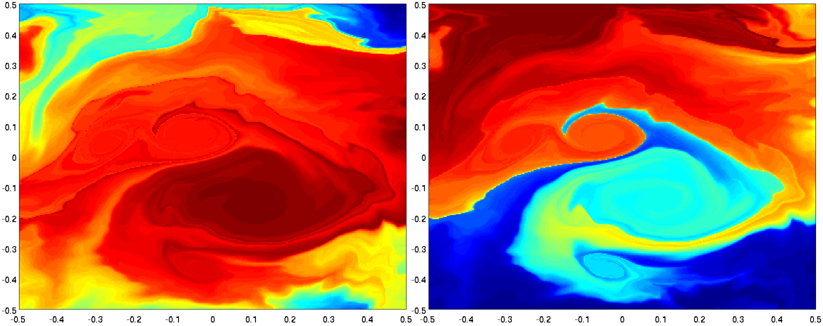
DAY=82



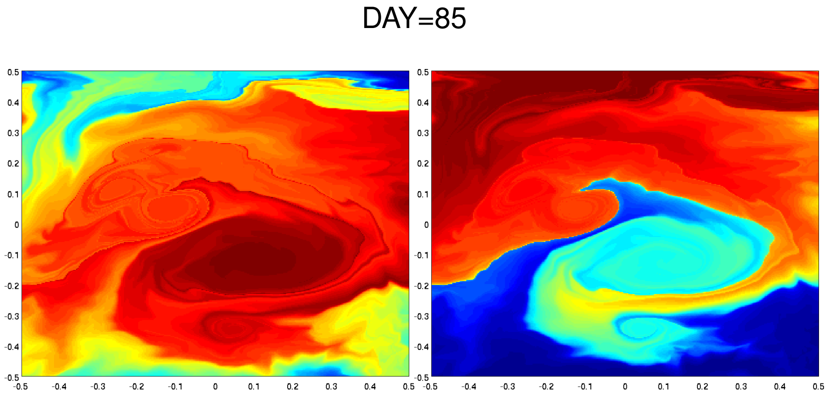
DAY=83



DAY=84

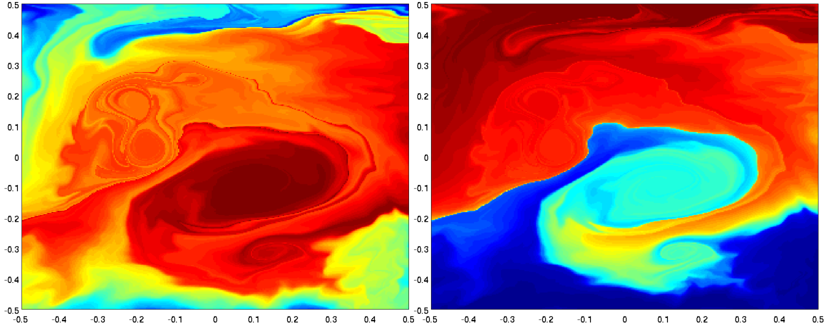


Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15



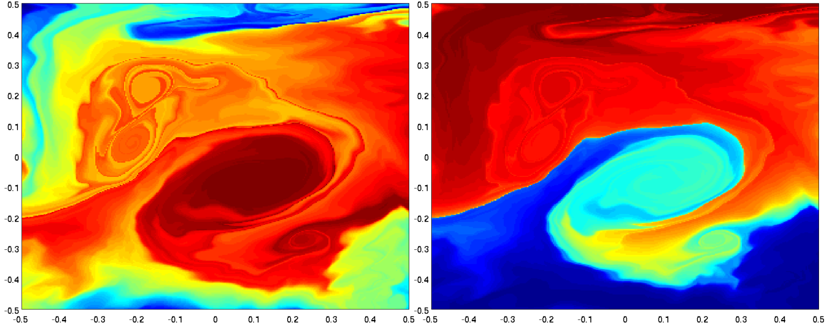
Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=86



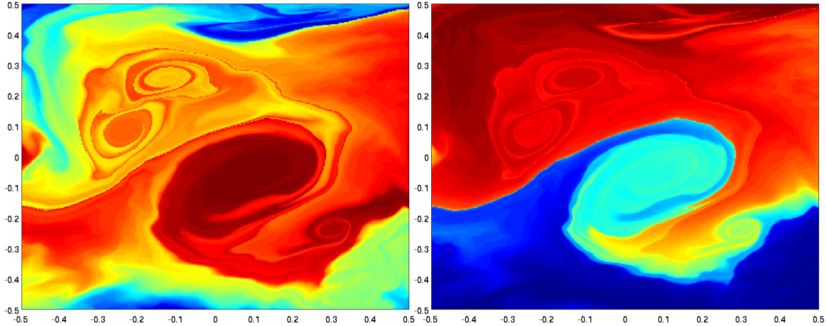
Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=87



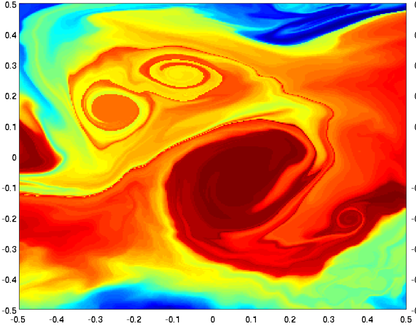
Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

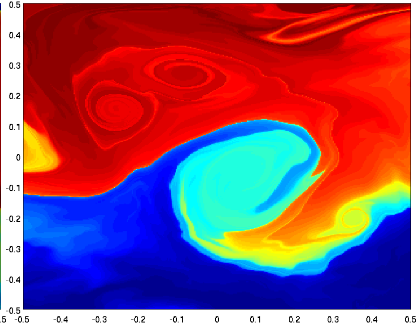
DAY=88



Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

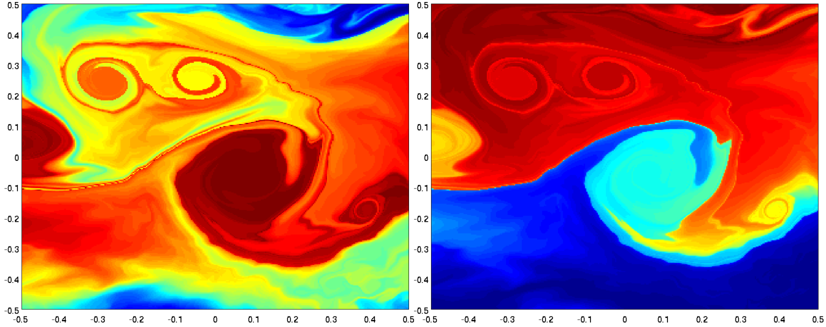
DAY=89



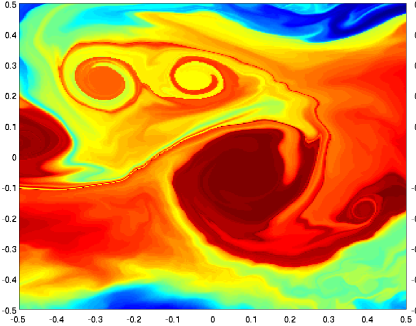


Time Dependence: Mid-depth/Shelf: $k_x = 0$, $k_y = 1/2$, Isopycnal 15

DAY=90



DAY=90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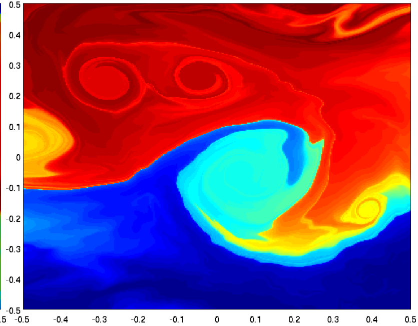
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5


0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
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-0.4
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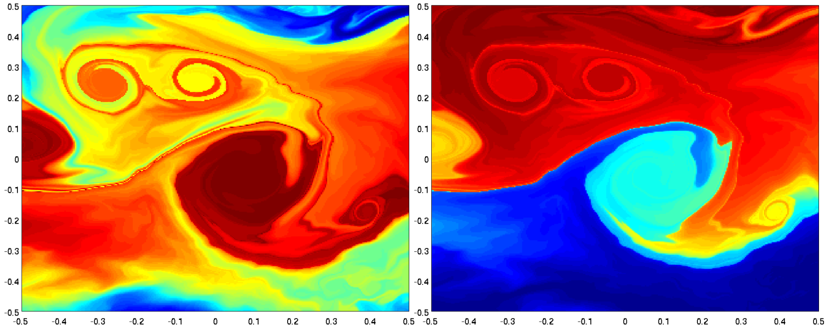
-0.1
0
0.1
0.2
0.3
0.4
0.5

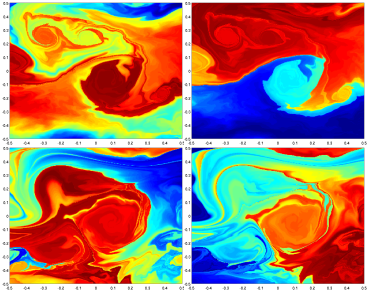
DAY=90



- Group by iso-values of single observable, $\overline{f_k(\mathbf{x}_0)}$
- Different kinematics - similar isovalues.
- Consider simultanenous vector of observables:

$$\mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})\}$$

- On averaging: $\bar{\mathbf{f}} : \mathbf{x}_0 \to \mathbb{R}^n$
- Color $\mathbf{x}_0$ by similarity in $\mathbb{R}^n$ (cluster algorithm)

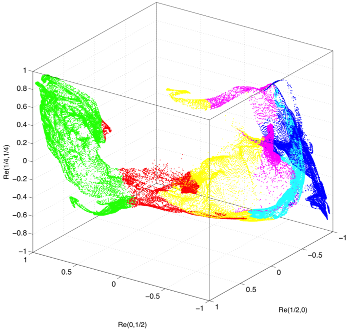


A Very Simple Example:

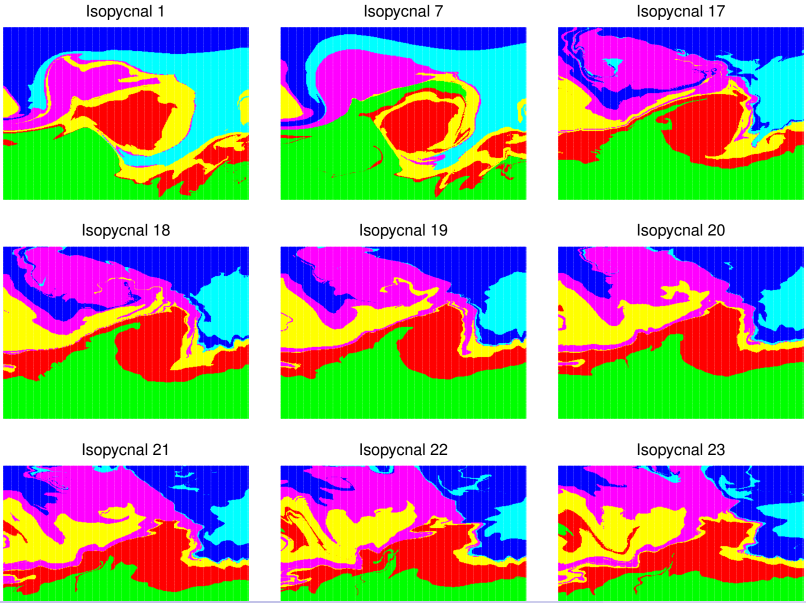
- Basis Set:

  $$f_{k,l} = \cos(kx)\cos(ly) + i\sin(kx)\sin($$

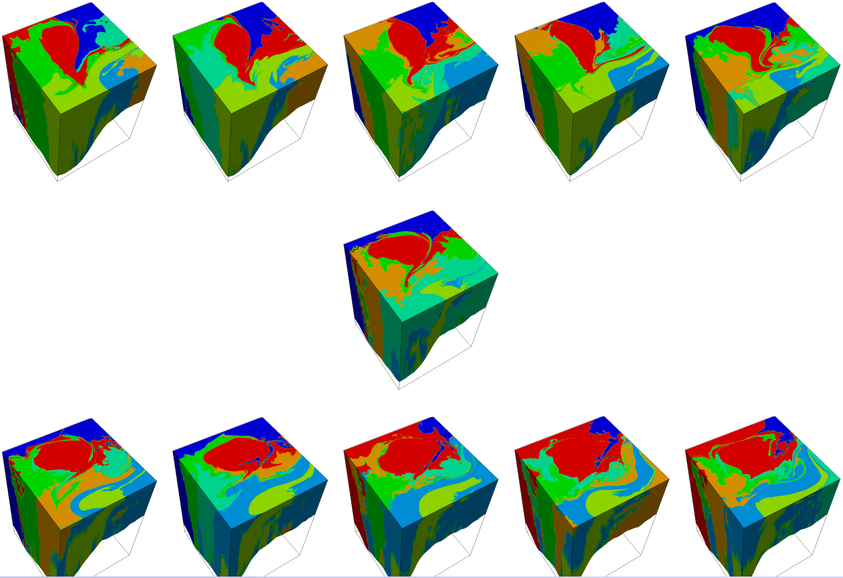
- $k = \{0, 1/4, 1/2\}$
- $(9 \times 10^4) \times 23$ points in $\mathbb{R}^{16}$
- Standard, quick, (dumb?) *k-means* clustering
- 6 clusters



6 Cluster Partition: Day 90



Partition: 3D Time Dependence



Summary:

- *Koopman* approach - simple, readily implemented, efficient means of identifying advective transport geometry in Eulerian frame.
    - Select: τ and nominal scales of interest through **f**.
    - Arbitrary dimensions. (2D/3D + 1)
    - Complex flows (arbitrary time/space dependence)

- Open question: Partition in open flows
    - Optimal basis sets? (Wavelets, ...)
    - Convergence? Within basis, across bases, with τ?
    - Finite-Time: Harmonic (phase) averaging.