Detecting Flow Coarse Patterns Using Lagrangian Averages

Marko Budisic
Igor Mezic

ONR MURI Ocean 3D+1
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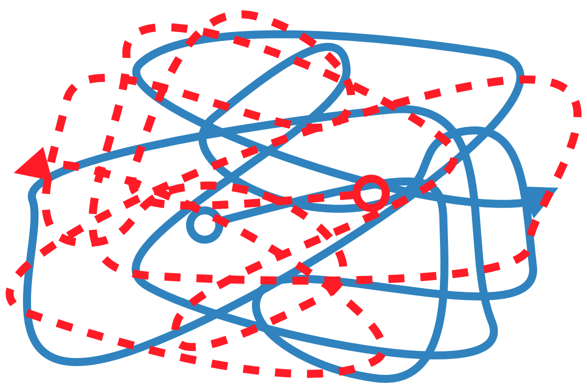
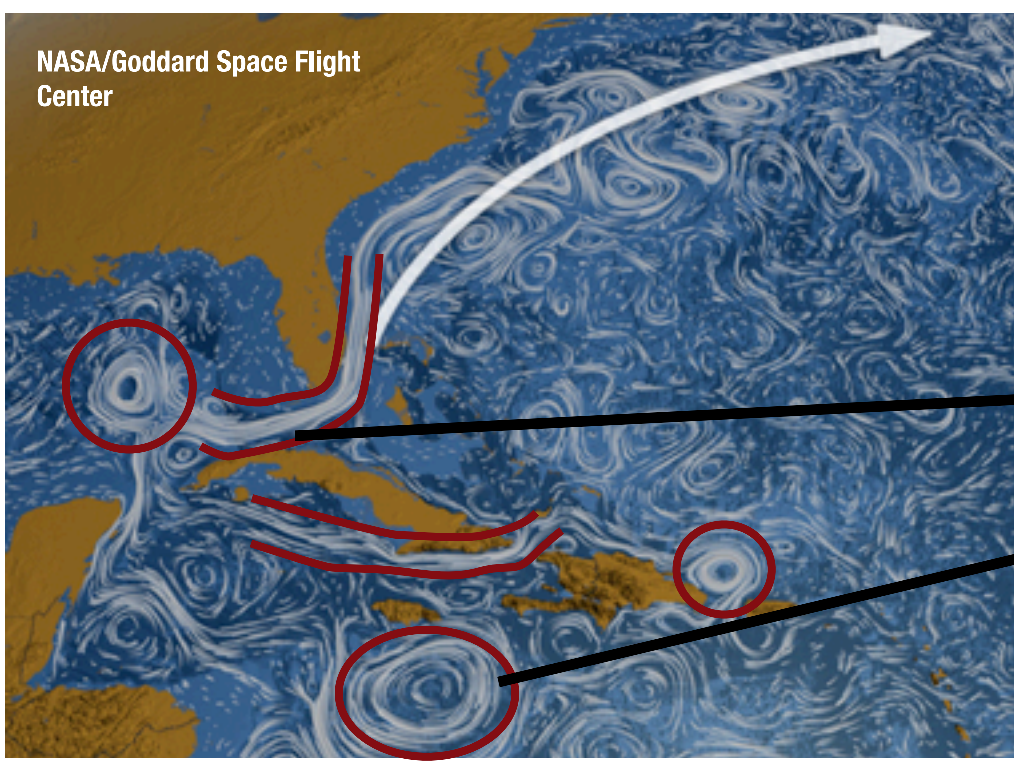
Goal: detecting coarse-grained patterns in fluid flows.



Comparison of tracer paths can be misleading:
Two trajectories in a mixing region
can never be aligned pointwise,
but **on average** they have the same behavior.

Approach: compare tracers according to averages of many different scalar fields.

Result: we can **quantify** when trajectories are **equal** on average, but also when they are **similar** on average.

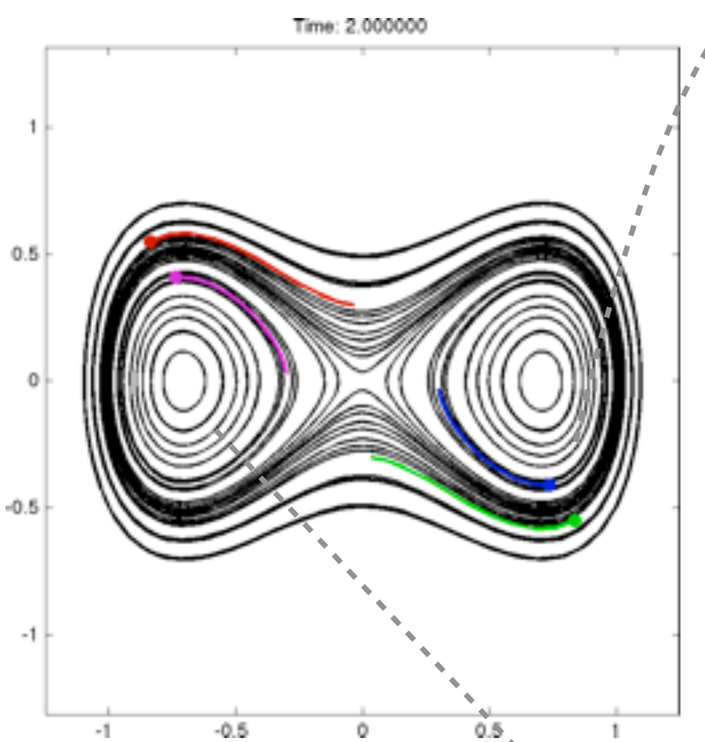


Layering: neighboring fluid parcels behave similarly

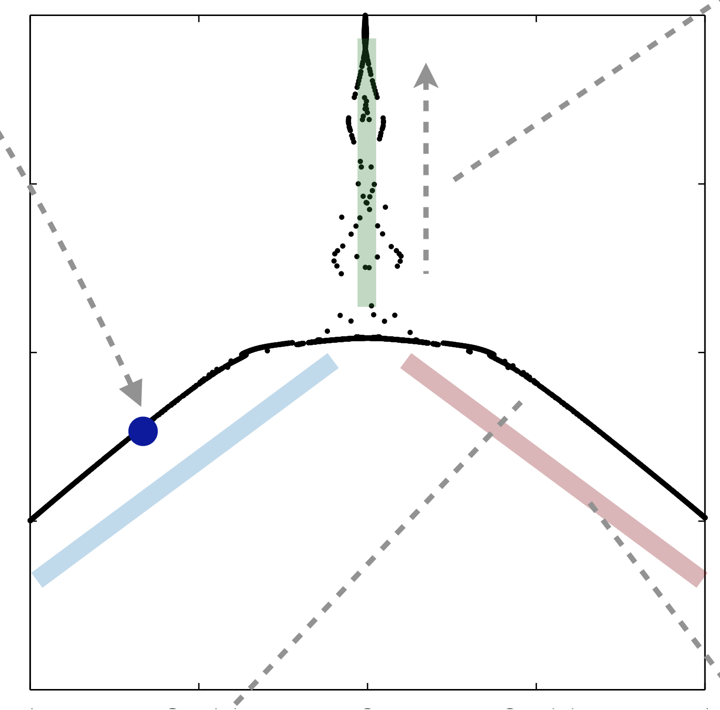
Ergodic quotient can be used to detect similarities in a multi-scale fashion.

Entire trajectories mapped to single points.

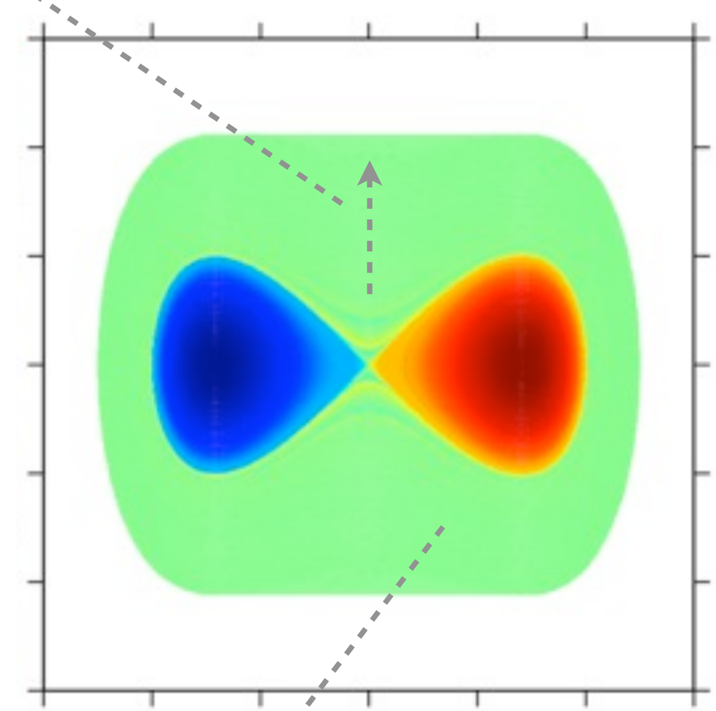
Axes in EQ act as generalized energies or stream functions.



Tracer paths



Ergodic Quotient



Colored initial conditions

Connected segments in EQ correspond to families of similar tracer paths.

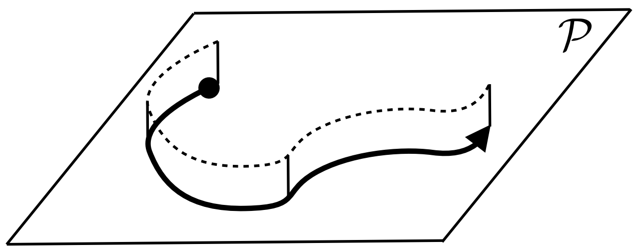
Coloring initial conditions according to membership in connected segments visualizes coarse patterns.

[Budisic, Mezic Physica D, 2012]

Ergodic quotient replaces trajectory curves by vectors of Lagrangian averages.

Curves:

$$\dot{x}_p = u(t, x_p), \quad x_p(0) = p$$
$$(p, t) \mapsto x_p(t)$$

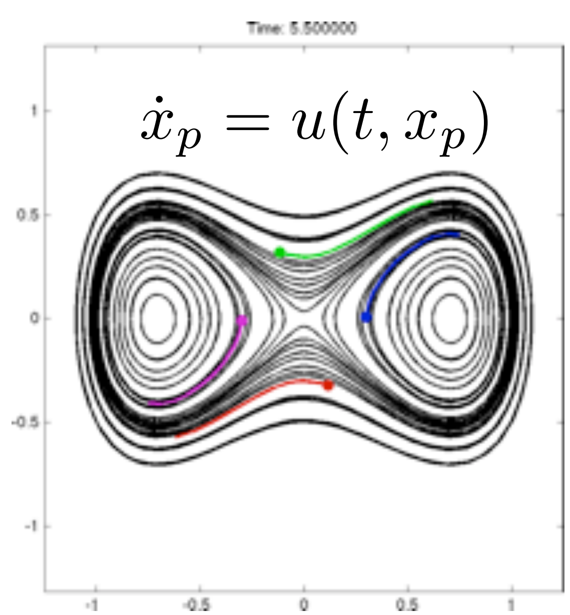


Lagrangian averages of scalar fields:

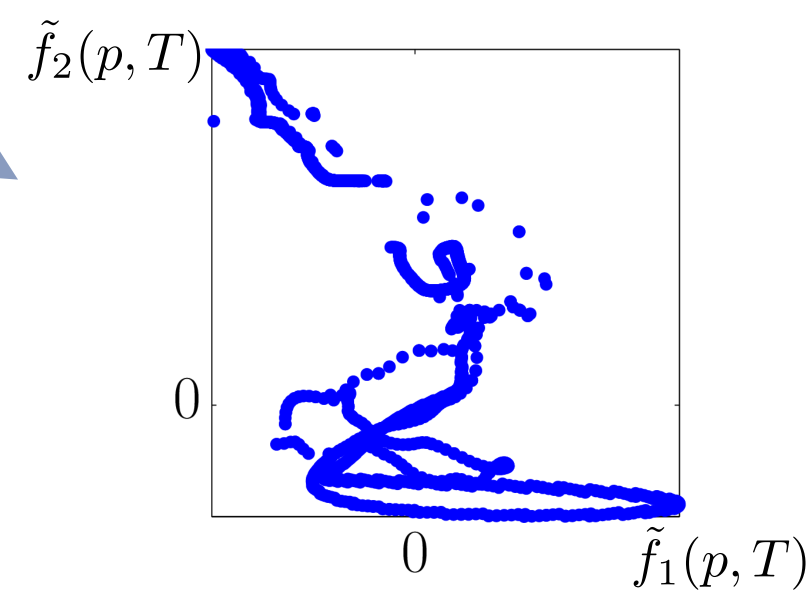
$$\tilde{f}(p, T) := \frac{1}{T} \int_0^T f_k(\tau, x_p(\tau)) d\tau$$

Ergodic quotient map is obtained by averaging a basis of continuous functions (scalar fields on the state space): $f_k(x) = e^{ik \cdot x}$

$$(p, T) \overset{\pi}{\mapsto} \begin{bmatrix} \tilde{f}_1(p, T) \\ \tilde{f}_2(p, T) \\ \vdots \end{bmatrix}$$



Representation of the tracer path portrait using averages of scalar fields.



The space of averages (finite-time quotient) naturally captures similarity.

Discrete topology (theorem):

Two state points p_1 and p_2
are in the same ergodic set iff $\pi(p_1, \infty) = \pi(p_2, \infty)$

minimal invariant set

Continuous topology: Sobolev space norm.

$$d_T(p_1, p_2)^2 = \sum_{k \in \mathbb{Z}^d} \frac{\left| \tilde{f}_k(p_1) - \tilde{f}_k(p_2) \right|^2}{(1 + |k|^2)^s}$$

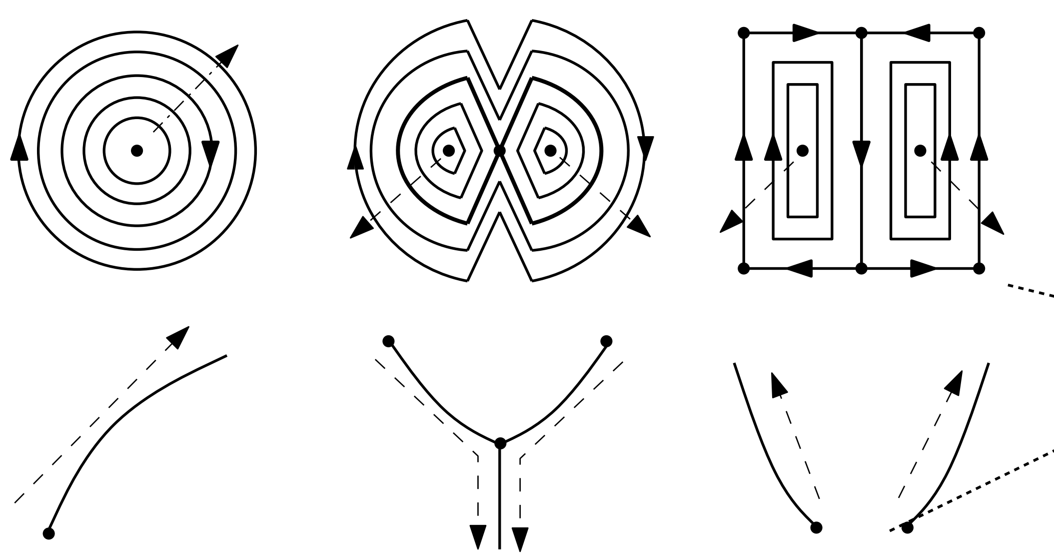
If scalar fields are chosen as Fourier harmonics, their averages are Fourier coefficients of averaging measures.

Acts as a low-pass filter:
de-emphasizes
small scale differences.

Ex.

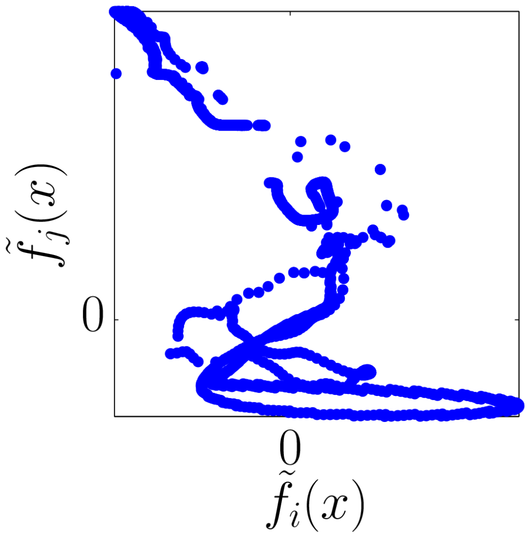
Tracer trajectories

Ergodic Quotient (in cts. topology)



Stagnation points on separatrices prevent ergodic quotient from connecting.

Diffusion maps are a nonlinear coordinate reduction that preserves intrinsic geometry of Ergodic Quotient (EQ).



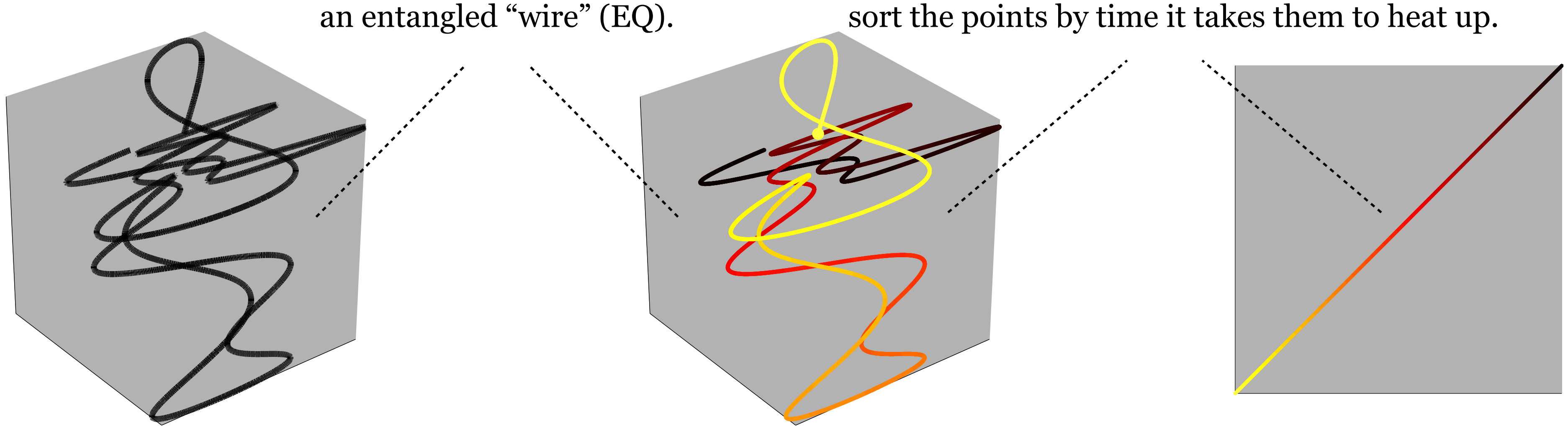
The **scalar fields used in averaging** were chosen regardless of dynamics, so they can yield a **high-dimensional space**.

The dimension of EQ can be **very low**, if the dynamics is simple, e.g., when there is only a **single gyre**, or a **single mixing region**.

Diffusion Maps:

A heat source is placed on an entangled "wire" (EQ).

To disentangle the "wire" (EQ), sort the points by time it takes them to heat up.

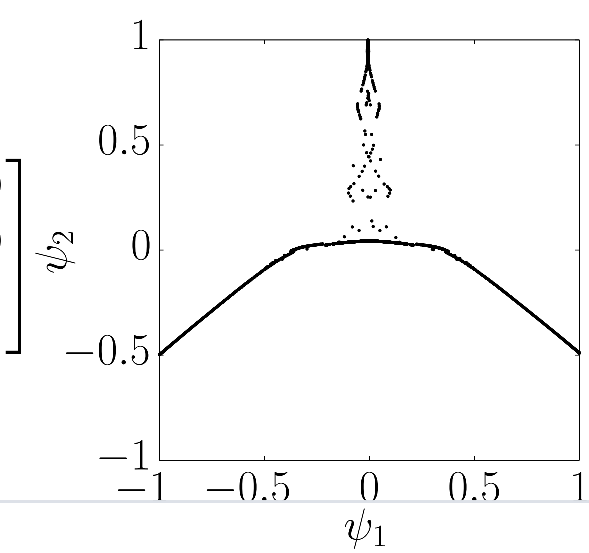


[Coifman, Lafon, ACHA, 2006]

Implementation requires only **deterministic matrix computations**.

Topology and geometry are preserved, e.g., a continuous line is still a line, but the **number of coordinates is greatly reduced**.

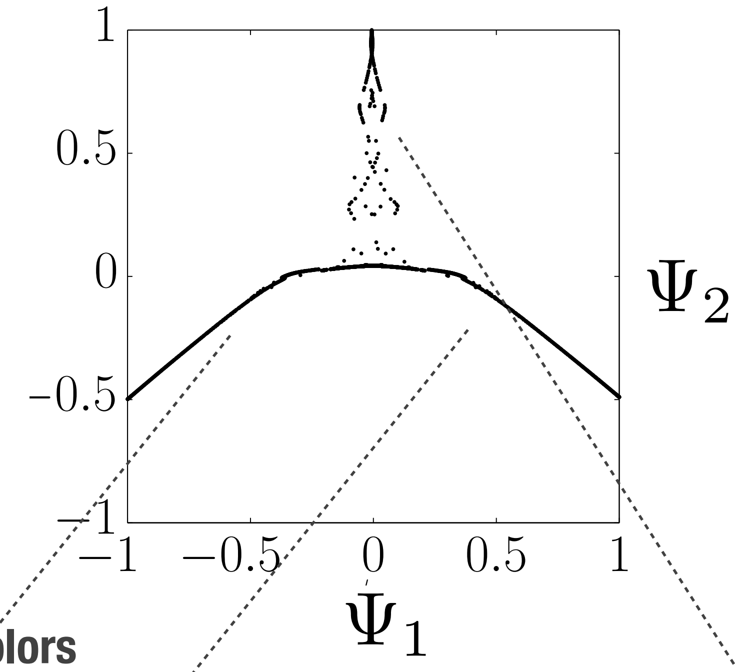
$$\underbrace{(p,T)}_{\text{Tracer paths}} \mapsto \underbrace{\begin{bmatrix}\tilde{f}_1(p,T)\\ \tilde{f}_2(p,T)\\ \vdots\end{bmatrix}}_{\text{Averaged fields}} \mapsto \underbrace{\begin{bmatrix}\Psi_1(p,T)\\ \Psi_2(p,T)\\ \vdots\end{bmatrix}}_{\text{Diffusion Coordinates}}$$



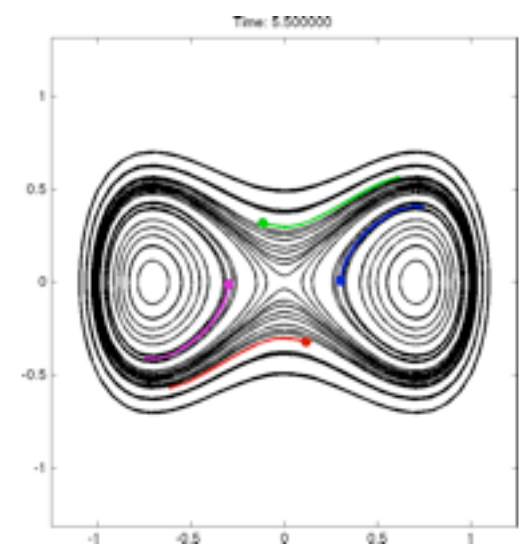
Coloring state space by values of dominant diffusion maps reveals large scale features.

Number of diffusion coordinates depends on complexity of dynamics, not dimension of the state space.

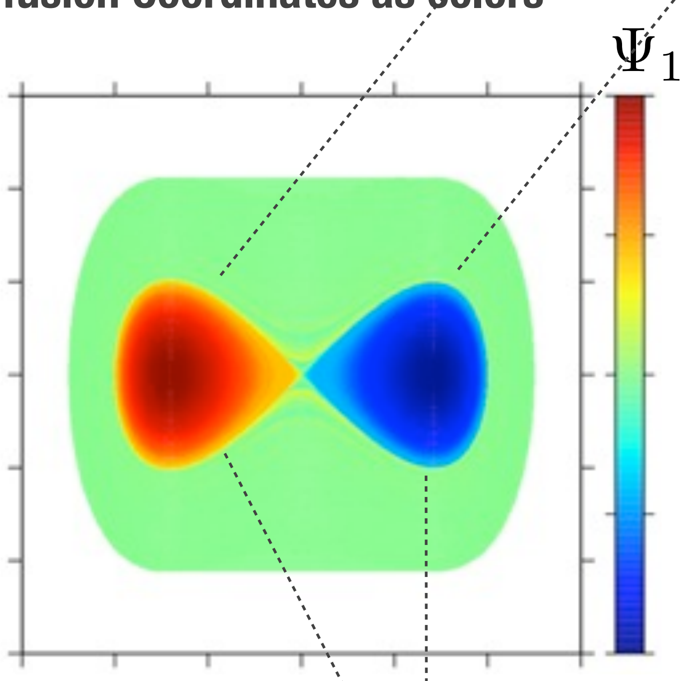
Diffusion Coordinates as axes

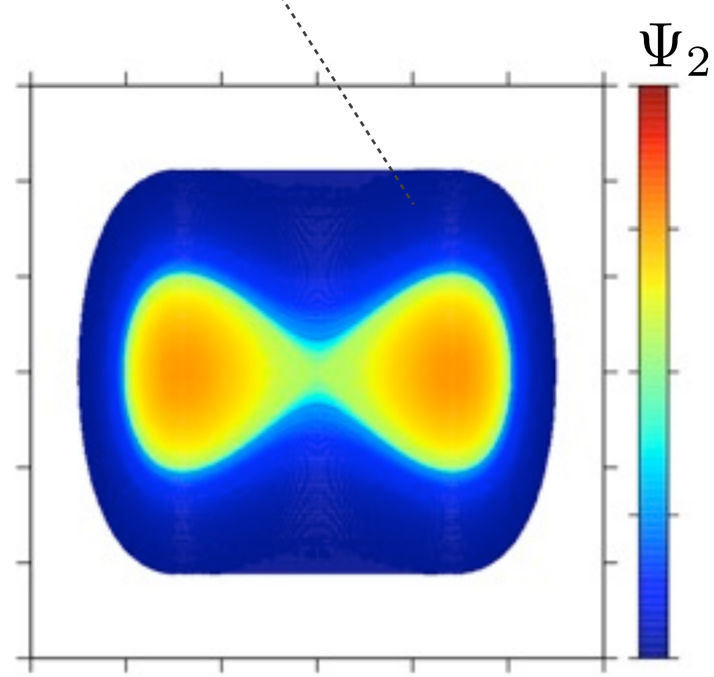


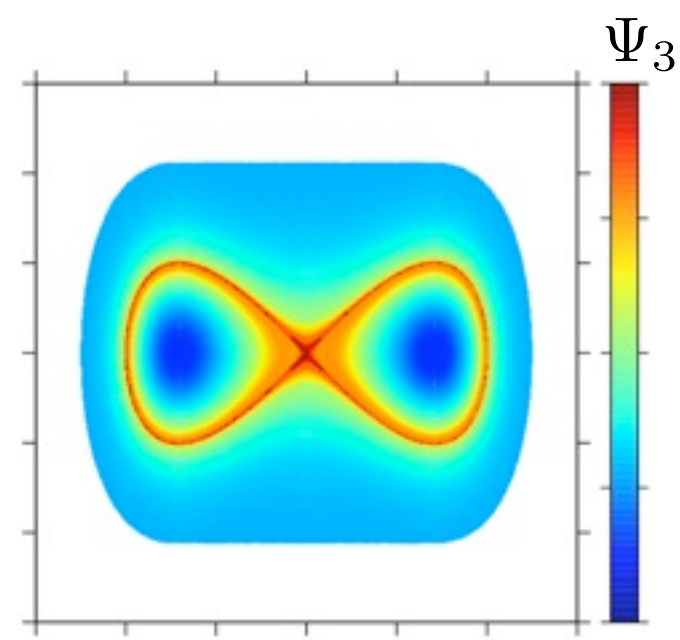
State Portrait



Diffusion Coordinates as colors







Different colors indicate there is no material transport between regions.

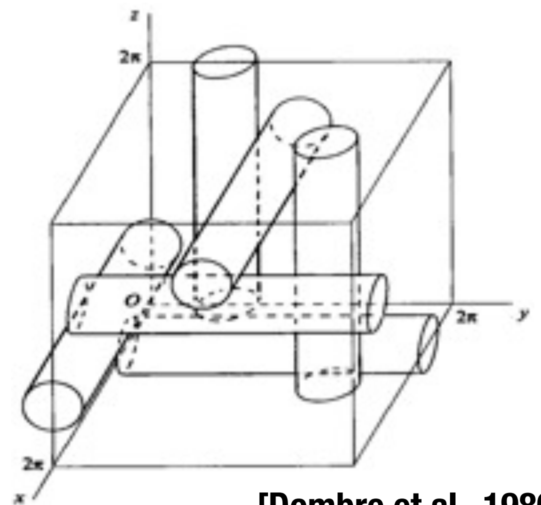
Coordinates of higher order distinguish between finer features.

Steady state 3D flow: ABC system.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A \sin z + C \cos y \\ B \sin x + A \cos z \\ C \sin y + B \cos x \end{bmatrix}$$

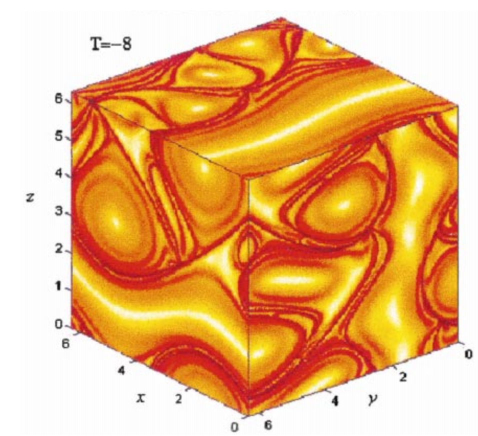
$$A^2 = 3, \; B^2 = 2, \; C^2 = 1$$

Analytic



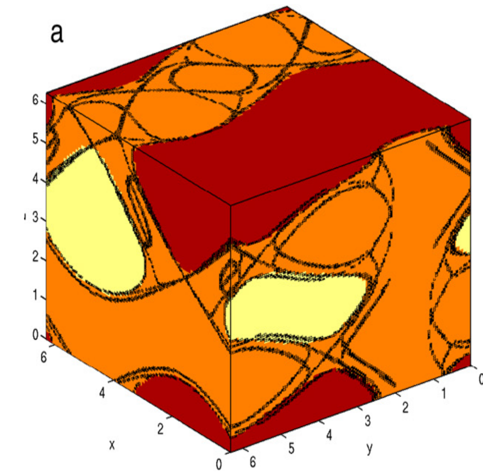
[Dombre et al., 1986]

LCS FTLE



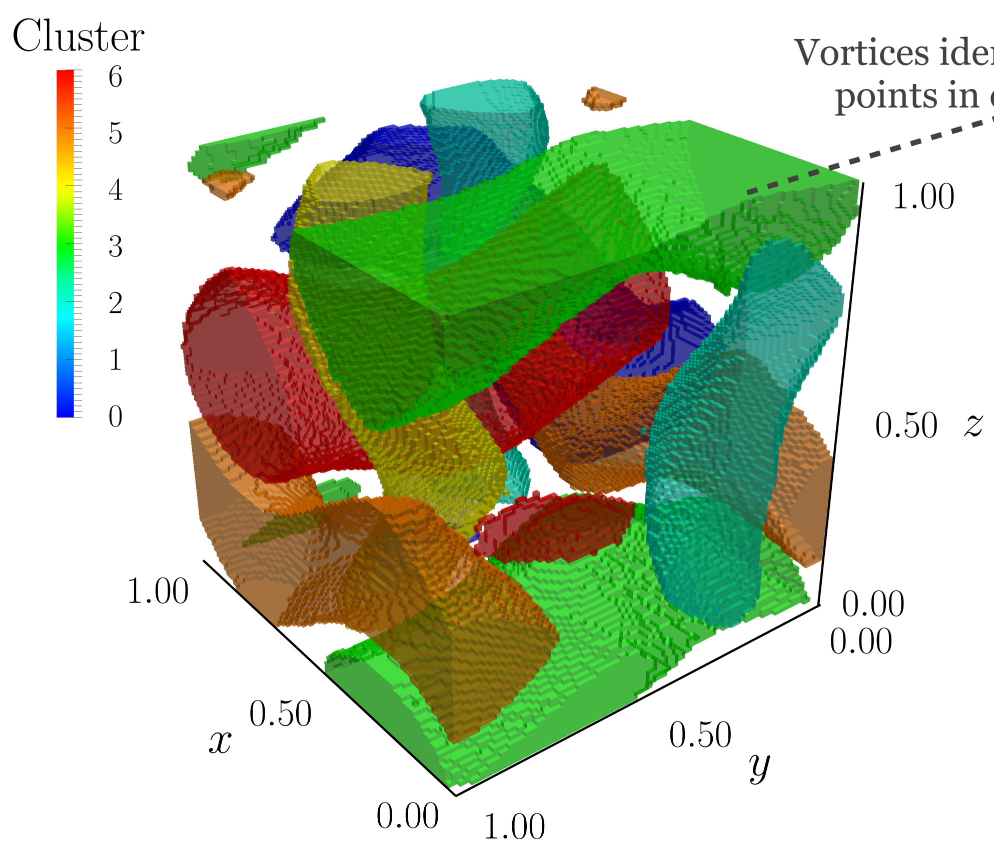
[Haller, 2001]

Almost-invariant sets



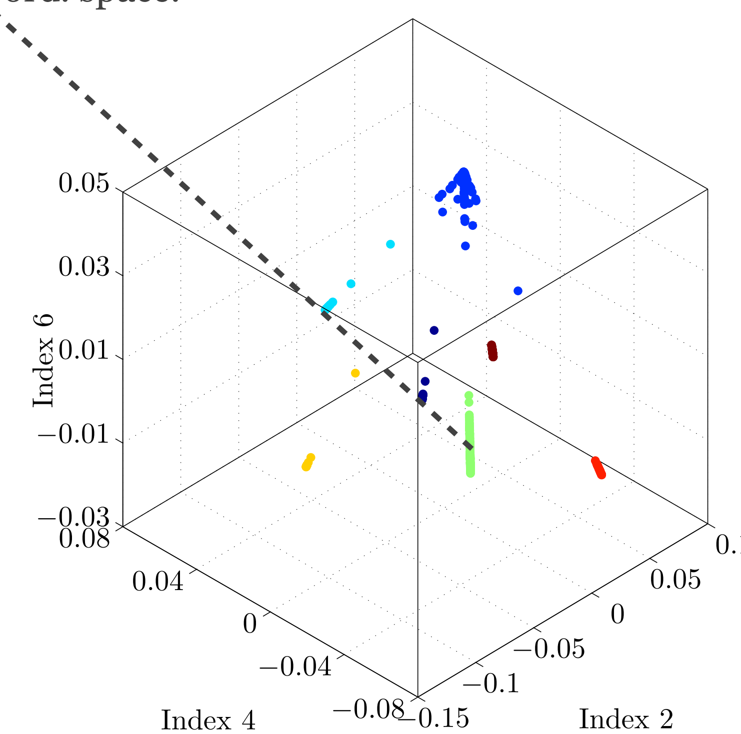
[Froyland et al., 2009]

State space coloring

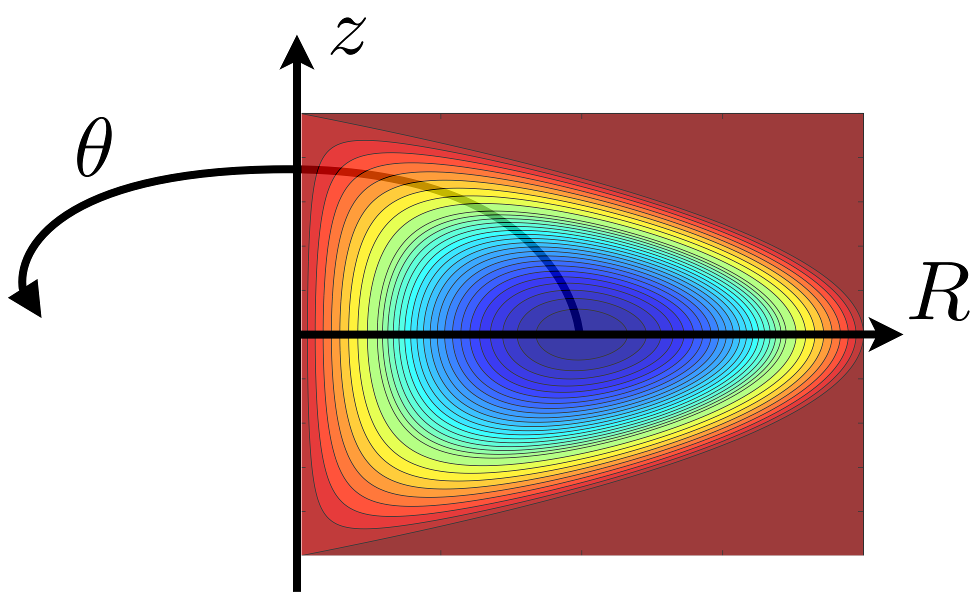


EQ in Diffusion Coordinates

Vortices identified as clusters of points in diff. coord. space.



Periodic 3D+1 flow: perturbed Hill's vortex ring



Hill's Vortex　　Perturbation

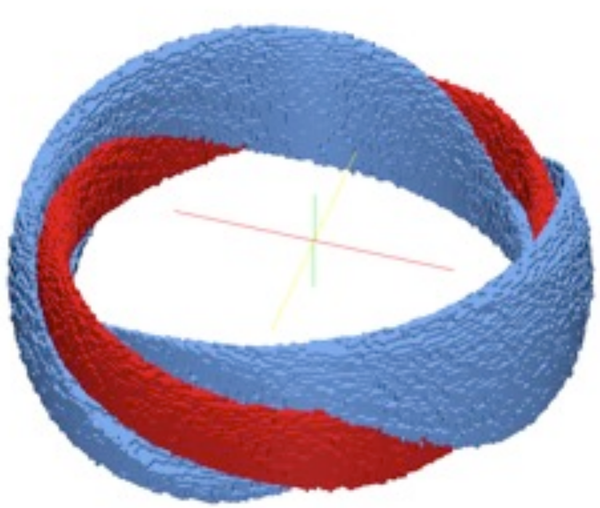
$$\begin{bmatrix}\dot{R}\\ \dot{z}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}2Rz\\ 1-4R-z^2\\ \frac{c}{2R}\end{bmatrix} + \varepsilon \begin{bmatrix}\sqrt{2R}\sin\theta\\ \frac{z}{\sqrt{2R}}\sin\theta\\ 2\cos\theta\end{bmatrix} \sin 2\pi t$$

Swirl

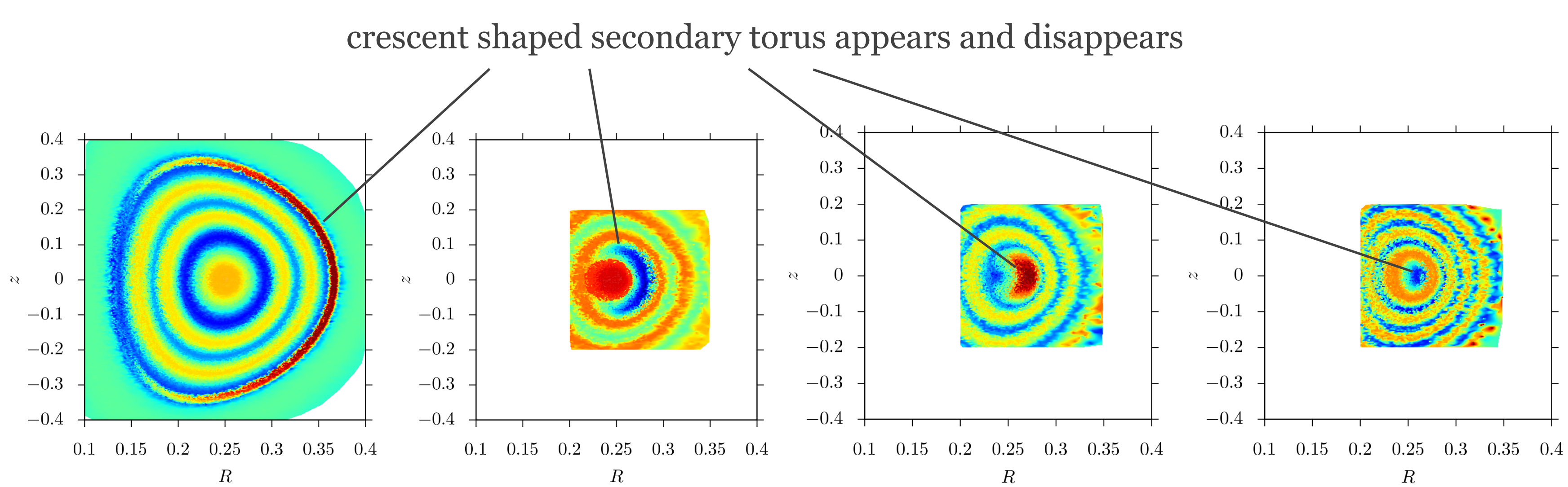
$$(R, z, \theta) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}$$

Unperturbed flow is Hamiltonian at each angular slice.
KAM behavior at small magnitude of swirl and perturbation.
For large perturbations, flow is difficult to study analytically.

Invariant tori in Poincaré section isolated using ergodic quotient:



New bifurcation identified:



(a) $\varepsilon = c = 0.280,\ k = 8$　(b) $\varepsilon = c = 0.349,\ k = 6$　(c) $\varepsilon = c = 0.352,\ k = 7$　(d) $\varepsilon = c = 0.355,\ k = 11$

Computational demands are driven by resolution desired, not dimension of the state space.

1. Select the subset of the basis of scalar fields.
2. Seed N initial conditions on the state space.
3. Integrate trajectories and averages along them.
4. Evaluate Sobolev distance matrix.
5. Compute diffusion coordinates.
6. Visualize.

The more initial conditions and scalar fields, the higher resolution of features.

Length driven by available data and application (finite time) or by a model (possibly infinite).

A numerical linear algebra computation: essentially an eigenvector computation for a matrix of size N x N.