Mean flow generated by circulation on a β-plane: An analogy with the moving flame experiment

By JOHN A. WHITEHEAD, JR., *Department of Physical Oceanography, Woods Hole Oceanographic Institution, Woods Hole, Mass. 02543, USA*

(Manuscript received May 31; revised version September 26, 1974)

ABSTRACT

A number of laboratory experiments are described in which water with a curved upper surface in a rotating basin exhibited prograde flows when stirred by stirrers which put no azimuthal torque upon the fluid. It is suggested that the flows were generated by Reynolds stresses of the circulating fluid, and that this is a general consequence of circulations on a β-plane. This is reinforced by an analogy between the equations of the moving flame experiment and the equations of flow on a β-plane. Implications upon atmospheric and oceanic flows are mentioned.

1. Introduction

Although it has become clear that momentum is transported by Reynolds stress in many oceanic and atmospheric phenomena (Starr, 1968), it has been difficult to develop theoretical tools which can adequately cope with the wide variety of processes which generate a net Reynolds stress. The most notable progress has been made in the momentum transfer properties of waves, whose dispersion properties allow one to make systematic developments of the wave equation. The same progress has not been made with eddies or other circulations which exist on many scales in the ocean atmosphere, as observed in the mesoscale by Polygon and MODE. In fact, only a few candidates for first principles for systematic momentum pumping by eddies are even presently hinted at, despite intense efforts to overcome severe theoretical problems. A central difficulty is that the Reynolds stres is capable of transporting momentum away from mean currents which have eddies in some instances such as in many turbulent mean flows, and yet it is capable of transporting momentum towards mean flows in other cases, such as in baroclinic instability, in the rotating flame experiment, and throughout the atmosphere.

Only a limited amount of laboratory observations have been made of the direct generation of a mean flow by eddies. Of those which have been conducted, the moving flame experiment has often been cited as the clearest example of such a process. In this experiment, a heater moving under a horizontal cylindrical annulus of liquid is observed to generate a mean flow in the liquid in a direction opposite to the direction of travel of the flame. This mean flow generation has been attributed theoretically to the phase lag in the overturning fluid due to the finite time it takes for vorticity and/or temperature to work its way into the fluid (Stern, 1959; Davey, 1967). A variety of analyses using this physical approach have been made by Schubert (1969), Kelley & Vreeman (1970), Malkus (1970), Schubert et al. (1970), Thompson (1970), Hinch & Schubert (1971), and Whitehead (1971), (1972), although Busse (1972) has found one exact solution where this is not so. This phase lag generates a net Reynolds stress. The Reynolds stress can have a significant cumulative effect, as is evidenced by the fact that flows three times faster than the "eddy propagation speed" were observed in experiments in liquid mercury (Whitehead, 1972).

It is not yet established whether the moving flame experiment has any direct relation to flows in nature. Schubert & Whitehead (1969) have suggested that this effect can generate a motion in the upper atmosphere of Venus, yet a lack of detailed observations, and a lack of a theory which is clearly applicable in the possible range of variables which are believed to apply to Venus has prevented a detailed comparison, in spite of a number of analyses of the problem (Malkus, 1970; Thompson, 1970). Stern (1971) has suggested the possibility of its playing a strong role in the dynamics of a tornado. Again, lack of observational comparison has prevented a detailed comparison.

Unfortunately, many of the eddies and waves in the oceans and atmosphere are associated with virtually inviscid dynamics, and phase lags are generated from processes such as wave dispersion, effects of the curvature of the earth, or the like, rather than diffusive phase lags. This has tended to weaken the usefulness of the moving flame experiments.

This paper is concerned with a new mechanism of mean flow generation which has the desirable features that it can be studied in the laboratory, is driven by physical processes which are believed to be important in both terrestrial and many planetary atmospheres, and is theoretically tractable. To be specific, an analogy between the moving flame equations and stirring in a β-plane is shown. Also, a variety of experiments will be described which suggest that such a mean flow is a general feature of flow on a β-plane when fluid is forced to new latitudes.

2. A physical reason for expecting a mean flow

First we will review the basic physical processes which cause the generation of the mean flow in the moving flame experiment. For illustrative purposes, we will consider a time-dependent, two-dimensional problem with the fluid bounded above and below by horizontal boundaries, and *with the fluid subjected to a* moving internal density field. We assume that the boussinesq approximation to the Navier-Stokes equation is valid, that is,

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\varrho_0}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g}\frac{\varrho}{\varrho_0} \tag{2}$$

and

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \nabla^2 T + Q(x, y, z, t) \tag{3}$$

where $\varrho = \varrho_0\ (1 - \alpha T)$ and Q is a source of heat in the fluid.

We will now invoke the assumptions that the problem only varies in the direction of gravity and the direction of the channel, that the density ϱ is a linear function of temperature, i.e., $\varrho = \varrho_0(1 - \alpha T)$, and finally that the Prandtl number of the fluid is very small. It can be shown that this last assumption is not necessary, in fact it is not done in most of the papers cited, but it will be done here for clarity in that it enables us to solve for the density field irrespective of the flow that is produced. Under these conditions the equations reduce to

$$u_x + v_y = 0 \tag{4}$$

$$u_t + uu_x + vu_y = -\frac{1}{\varrho_0}p_x + \nu\nabla^2 u \tag{5}$$

$$v_t + uv_x + vv_y = -\frac{1}{\varrho_0}p_y + \nu\nabla^2 v - \frac{g\varrho}{\varrho_0} \tag{6}$$

where subscripts denote a partial derivative and $\mathbf{x}$ and $\mathbf{y}$ are the direction of the axis of the channel and gravity, respectivey. We assume here that the density field ϱ is a known solution to (3) and is moving uniformly along the channel in time, i.e.

$$\varrho = \varrho(x - ut, y) \tag{7}$$

Eq. (4)–(6) are combined by cross differentiating (5) and (6) and subtracting and defining a potential which satisfies (4) as $u = \psi_y$, $v = -\psi_x$. This yields the formula:

$$\nabla^2\psi_t - \nu\nabla^4\psi = \frac{g\varrho_x}{\varrho_0} + J(\psi, \nabla^2\psi) \tag{8}$$

We now will transform to a coordinate system moving with velocity U, by defining

$x' = x - Ut$, and rewrite (8) as

$$-U\nabla^2\psi_{x'} - \nu\nabla^4\psi = \frac{g\varrho_{x'}}{\varrho_0} + J(\psi, \nabla^2\psi) \tag{9}$$

where all derivatives are with respect to x'.

It is my purpose here to point out that certain flows in a β-plane obey equations very similar to (9) and therefore contain almost the identical physics. This can be seen most clearly by inspection of the following model, reminiscent of Stommel's (1948) model of westward intensification. We look at the equations of depth-averaged motion on a β-plane.

$$u_x + v_y = 0 \tag{10}$$

$$uu_x + vu_y + (f_o + \beta y)\, v = -\frac{1}{\varrho} p_x + \nu \nabla^2 u + F' \tag{11}$$

$$uv_x + vv_y - (f_0 + \beta y)\, u = -\frac{1}{\varrho} p_y + \nu \nabla^2 v + F'' \tag{12}$$

where plus y is north.

Taking $\partial/\partial y$ of (11), $\partial/\partial x$ of (12), subtracting, and defining velocities in terms of a potential ψ which satisfies (10) as follows: $u = \psi_y$, $v = -\psi_x$ we get

$$(-\beta\psi_x - \nu\nabla^4\psi) = F'_y - F''_x + J(\psi, \nabla^2\psi) \tag{13}$$

The central point of this section is that the β-effect generates a governing equation which is identical to eq. (9) in the moving flame experiment except for one laplacian operator. A central feature in both equations is the asymmetry about the x-direction introduced by the first order derivative. Some conseqeunces of this asymmetry are dealt with in the literature previously cited. Briefly, in all cases except for a few exact solutions, the first order derivative in x tilts the streamlines so that eastward momentum is fed from the boundaries into the interior.

3. Laboratory observations

It was desired to conduct a number of experiments in which a rotating mass of water of variable depth was stirred with a stirrer which puts no direct force in the direction of rotation. The 2 meter diameter turntale in the hydrodynamics laboratory at the Woods Hole Oceanographic Institution was filled with 10 cm of water, and was then spun up to a period of revolution of 4 seconds. In the absence of air drag, the top surfce would have a depth above the flat bottom of the following form: $h = 2.934 + 0.00126r^2$ where r is distance

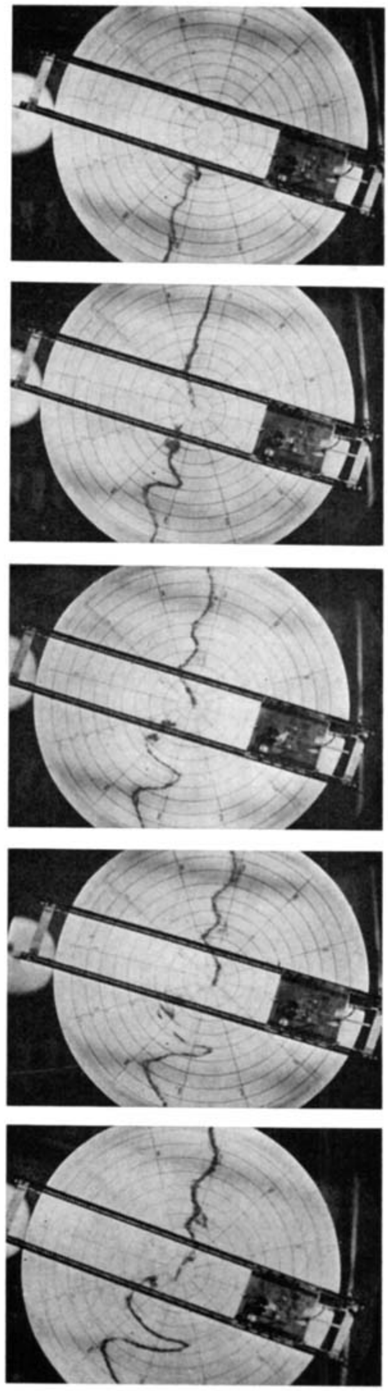

Fig. 1. Dye streak photographs of prograde flow in the rotating fluid with strong agitation. ($\varepsilon_p = 0.106$) Pictures are 16 seconds apart. The stirrer is under the rectangular plate in the 4 o'clock position.

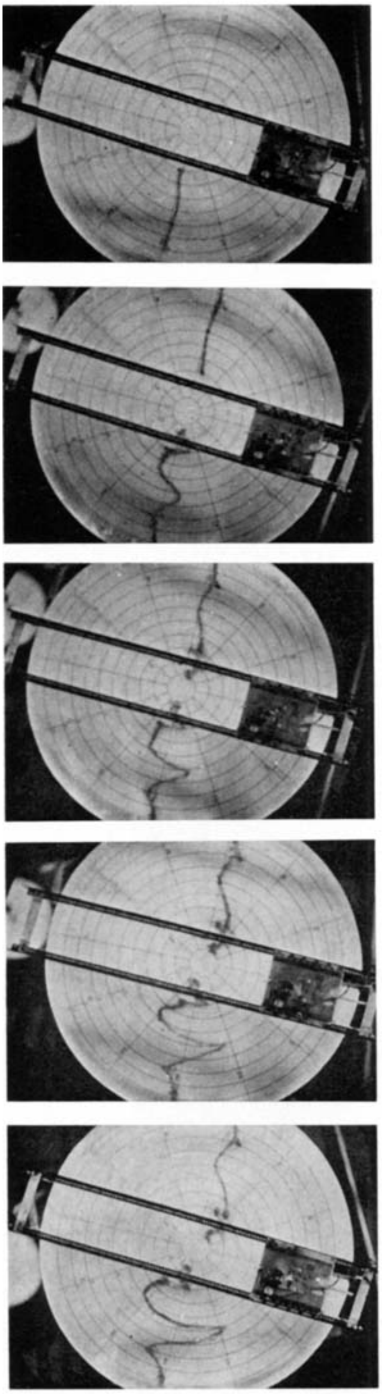

Fig. 2. Dye streak photographs of prograde flow in the rotating fluid with gentle agitation ($\varepsilon_p = 0.0148$) and with the tank covered. Pictures are 32 seconds apart.

from the center and h is depth of the fluid (in centimeters).

The first stirrer consisted of a horizontal disk 20 cm in diameter connected to a scotch yoke of variable amplitude driven by a variable speed motor. When the disk was centered 60 cm from the center of the turntable and oscillated up and down in the 6.72 cm deep water at periods from 4 to 40 seconds, and with a variety of amplitudes, a pronounced prograde flow was observed due west of the plunger, between a radius of 50 to 60 cm from the center (the marked circles are 10 cm apart). This jet sometimes stretched completely around the tank. A retrograde flow was observed in other regions of the tank. Such a flow is shown in the dye-streak photos in Fig. 1. In order to measure this flow more carefully, the table was covered with sheets of vinylidene chloride (saranwrap), with one slitlike opening left 90° west of the plunger so that a saturated solution of potassium permanganate could be injected as a flow tracer. This covering reduced the retrograde flow to one revolution in 10^6 table revolutions, which was approximately one order of magnitude slower than the slowest flows measured. Fig. 2 shows a slow flow under such conditions. Velocity was measured by timing the traverse time for dye to travel between 90°W and 60°W (the plunger is at 0°) a distance of 31.4 cm. The data were taken for various plunger periods from 4.1 seconds up, and for plunger amplitudes of 1.75 cm, 3.5 cm, and 5.2 cm. Fig. 3 shows the data points obtained. Both ordinate and abscissa are presented as Rossby number, the Rossby number of the plunger defined as $\varepsilon_p \equiv AP_t/RP$ where A is amplitude of the plunger motion, P_t is period of one table revolution, R is radius of the plunger, and P_p is period of one plunger cycle. The Rossby number of the current was defined as $\varepsilon_c = U/\Omega L$ where U is the measured velocity of the jet, Ω is angular rotation rate of the table, and L is a half width of the jet, which was placed at 10 cm for all data even though there was slight variation about that value.

Flow rate of the jet increased sharply with velocity of the plunger for low plunger speeds. It is not inconceivable that this obeys a quadratic power law (a dashed line with a slope of 2:1 is inserted for comparison). At greater plunger speeds the flow rate of the jet

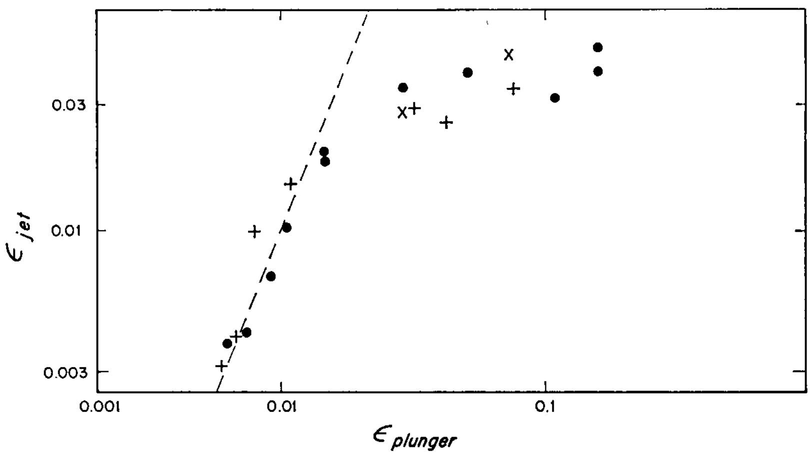

Fig. 3. Rossby number of the flow as a function of Rossby number of the plunger for experiments with a covered tank. Amplitude (peak to peak): X −1.75 cm, −3.5 cm, +5.20 cm. A dashed line with a slope of 2:1 is inserted for comparison.

is relatively insensitive to the plunger speed. During the course of these experiments it was obvious that in this latter case there was extensive turbulence in the fluid, while in the former case there was virtually no turbulence. The exact nature of the flow field around this plunger is poorly understood and a more detailed experimental investigation and an accompanying theoretical analysis is planned.

However, this oscillating plunger is not the only mixer that generates a mean flow, because a variety of other mechanisms were used. Fig. 4 shows dye being swept by a mean flow generated by air bubbles rising from a pipe with 20 holes located 60 cm from the center, at approximately a 4 o'clock position. The flow was qualitatively very similar to the flow generated by the plunger.

A similar flow was generated by moving a vertical sheet of plastic which is 30 cm long in and out, with an amplitude (peak to peak) of about 9 cm. Although the flow was not as pronounced as in the other two cases, it was qualitatively similar and was prograde at the "latitude" of the sheet, and retrograde north and south of it.

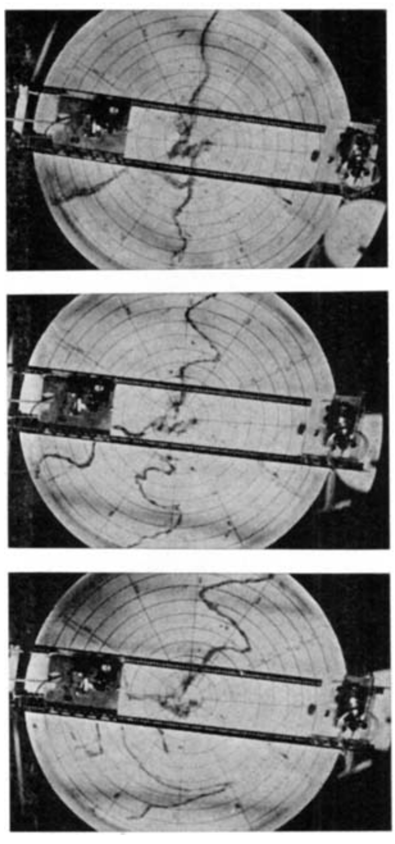

Fig. 4. Dye streak photographs of prograde flow generated by air bubbles from a header located at approximately a 4 o'clock position.

Comments upon applications

There exists in the literature a number of theoretical and numerical studies which show clear evidence of momentum flux on spherical shells or on β-planes. It is the intent here to explain why this is a general feature of stirring on a β-plane. A whole class of theoretical studies of the baroclinic instability of a zonal current have shown that the eddies, which manifest the instability, generate a Reynolds stress which intensifies the current at both infinitesimal (Phillips, 1954) and at finite amplitude (Pedlosky, 1970). However, such a transport does not seem to be an exclusive feature of such an instability. Thompson (1971) presented a simple argument which showed that any energy source which radiated Rossby waves northward and southward would be characterized by a convergence of eastward (prograde) momentum. An accumulation of eastward momentum has also been predicted by Nickel (1969), Busse (1970), and Yavorskaya et al. (1972) for the case of convection in a spherical shell heated from within. Lastly, Stern (1971) showed that an energy source in a uniformly rotating plane would radiate rotational momentum and would thus be characterized with an accumulation of antirotational momentum.

We close by pointing out that the streamlines of winddriven flow in a β-plane in Stommel's original paper (Stommel 1948) exhibit a pronounce eastward tilt in mid-latitudes which generates a Reynolds stress which would accumulate eastward momentum in the interior of the β-plane ocean. Although this effect is small in the parameter range suggested in that analysis, many geophysical problems are in a parameter range where this effect is conceivably large.

Acknowledgements

Thanks are due to Professor V. P. Starr and his students for many interesting discussions and suggestions. Thanks are also due to Dr Rory Thompson for compelling encouragement. Mr Robert Frazel assisted in the experiments and deserves credit for the excellent photographs. Research supported by the Oceanography Section, National Science Foundation, Grant GA-35447. Contribution number 3222 of the Woods Hole Oceanographic Institution.

REFERENCES

Busse, F. H. 1970. Differential rotation in stellar convection zones. *Ap. J. 159*, 629–639.

Busse, F. H. 1972. On the mean flow induced by a traveling thermal wave. *J. Atmos. Sci. 8*, 1423–29.

Davey, A. 1967. The motion of a fluid due to a moving source of heat at the boundary. *J. Fluid Mech. 29*, no. 1, 137–150.

Hinch, E. J. & Schubert, G. 1971. Strong streaming induced by a moving thermal wave. *J. Fluid Mech. 47*, 291–304.

Kelley, R. E. & Vreeman, J. D. 1970. Excitation of waves and mean currents in a stratified fluid due to a moving heat source. *Angew. Math. Phys. 21*, 1.

Malkus, W. V. R. 1970. Hadley-Halley circulation on Venus. *J. Atmos. Sci. 27*, 529–535.

Nickel, G. H. 1969. *Solar Physics 10*, 472–475.

Pedlosky, J. 1970. Finite-amplitude baroclinic waves. *J. Atmos. Sci. 27*, 15–30.

Phillips, N. A. 1954. Energy transformations and meridional circulations associated with simple baroclinic waves in a two-level, quasi-geostrophic model. *Tellus 6*, 273–826.

Schubert, G. & Whitehead, J. A. 1969. Moving flame experiment with liquid mercury: possible applications for the Venus atmosphere. *Science 163*, 71, 72.

Schubert, G. 1969. High velocities induced in a fluid by a traveling thermal source. J.A.S. *26*, 767–770.

Schubert, G., Young, R. E. & Hinch, J. 1970. Prograde and retrograde motion in a fluid layer: Consequences for thermal diffusion in the Venus atmosphere. *J. Geophys. Res. 76*, 21219.

Starr, V. P. 1968. Physics of negative viscosity phenomena. McGraw-Hill, New York.

Stern, M. E. 1959. The moving flame experiment. *Tellus 11*, 175–179.

Stern, M. E. 1971. Generalizations of the rotating flame effect. *Tellus 23*, 122–128.

Stommel, H. 1948. The westward intensification of winddriven ocean currents. *Trans. Amer. Geophys. Union 29*, 202–206.

Thompson, R. (1970). Venus's general circulation is a merry go round. *J. Atmos. Sci. 27*, 1107–1116.

Thompson, R. O. R. Y. 1971. Why there is an intense eastward current in the North Atlantic but not in the South Atlantic. *J. Phys. Oceanog. 1*, 235–237.

Whitehead, J. A. 1971. The generation of mean flows by a negative Reynolds stress. *Environmental and Geophysical Heat Transfer* (ed. C. J. Cremers, F. Kreith and J. A. Clark). Am. Soc. Mech. Eng., New York.

Whitehead, J. A. 1972. Observations of rapid mean flow produced in mercury by a moving heater. *Geoph. Fluid Dyn. 3*, 161–180.

Yavorskaya, I. M., Keller, B. S. & Mel'nikova, N. S. 1972. Scientific Programme, 13th International Congress of Theoretical and Applied Mechanics, Moscow, USSR.

СРЕДНИЙ ПОТОК, ГЕНЕРИРУЕМЫЙ ЦИРКУЛЯЦИЕЙ НА β-ПЛОСКОСТИ: АНАЛОГИЯ С ЭКСПЕРИМЕНТОМ С ДВИЖУЩИМСЯ ПЛАМЕНЕМ

Описывается ряд лабораторных экспериментов, в которых вода и искривленной верхней поверхностью во вращающемся бассейне проявляет прямое течение, когда перемешивается мешалками, не накладывающими на жидкость усилия в азимутальном направлении. Предполагается, что потоки генерируются напряжениями Рейнольдса циркулирующей жидкости и что это является общим следствием циркуляции на β-плоскости. Это утверждение усиливается аналогией между уравнениями, описывающими эксперимент с движущимся пламенем и уравнениями потока на β-плоскости. Упоминаются следствия этого рассмотрения для атмосферных и океанических течений.