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THE KORTEWEG-deVRIES EQUATION FROM LABORATORY CONDUIT AND MAGMA MIGRATION EQUATIONS

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Abstract. Wave equations that govern porous flow in a matrix and analogous equations for fluid conduits can be reduced to the Korteweg-deVries equation in the limit of small perturbations.

Introduction

Recently, two new fluid flow problems have exhibited solitary waves which have soliton-like behavior. That is, the waves are conserved upon collisions with other waves. The first problem arose from an attempt to understand how melt could travel around solid crystalline grains in a rock that was reaching its melting temperature. In numerical experiments with equations describing porous flow in a deformable matrix (Scott and Stevenson, 1984; Richter and McKenzie, 1984), the migrating fluid readily adopted solitary wave behavior. The second problem came from an attempt to understand how low-viscosity materials could buoyantly rise through material with much higher viscosity. It was found that the low viscosity fluid could rise through vertical cylindrical conduits or pipes (Whitehead and Luther, 1974). Recently Scott, Stevenson and Whitehead (1986) and Olson and Christenson (in preparation) have not only observed solitary waves on the walls of the conduit, but Scott et al. report soliton-like collisions and a clear analogy between the porous flow problem and the conduit problem. More laboratory observations of solitary wave collisions in the conduits are being reported elsewhere (Whitehead, in preparation). However, it has not been shown analytically that these equations possess exact soliton behavior.

Derivation of the Korteweg-deVries Equation

The Korteweg-deVries (KdV) equation is the simplest equation for solitons (Newell, 1985). This note shows that it can be derived from equations (4) and (5) of Scott, Stevenson and Whitehead (1986) which describe viscous conduit flow. Rewriting their (4) and (5) using $A = \pi a^2$ gives

$$\frac{\partial A}{\partial t} = -\frac{\partial u}{\partial z} \tag{1}$$

$$u = \frac{A^2}{8\pi\eta^1}\left[g\Delta\rho + \eta^s \frac{\partial}{\partial z}\frac{1}{A}\frac{\partial u}{\partial z}\right] \tag{2}$$

where $a(z)$ is the radius of the conduit, A is cross-sectional area, u is volumetric flux up the conduit, z is vertical coordinate, t is time, η^1 is viscosity of the rising fluid, η^s viscosity of the host fluid, g is acceleration due to gravity, and $\Delta\rho$ is the density difference between the two fluids. After Olson and Christensen we will use the following scales for volume flux, length and time scales

$$u_o = \frac{g\Delta\rho A_o^2}{8\pi\eta^1}$$

$$L = \left[\frac{\eta_s A_o}{8\pi\eta^1}\right]^{1/2}$$

$$T = \frac{1}{g\Delta\rho}\left[\frac{8\pi\eta^1\eta^s}{A_o}\right]^{1/2} .$$

where A_o is the steady conduit area, and u_o is the steady flux. Defining the dimensionless variables

$$u = u_o Q$$

$$A = A_o B$$

$$z = L\zeta$$

$$t = T\tau ,$$

(1) and (2) become

$$\frac{\partial B}{\partial \tau} + \frac{\partial Q}{\partial \zeta} = 0, \tag{3}$$

$$Q = B^2 + B^2 \frac{\partial}{\partial \zeta}\frac{1}{B}\frac{\partial Q}{\partial \zeta} . \tag{4}$$

We now expand the dependent variables in a power series

$$B = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \ldots \tag{5}$$

$$Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + \ldots \tag{6}$$

where the expansion parameter $\epsilon << 1$. Here ϵ is a measure of the departure of the conduit area from the basic conduit state. This implies that departure from the basic conduit state, B_0, $Q_0 = 1$, is small. The stretched coordinate system

$$\sigma = \epsilon^{3/2}\tau \tag{7}$$

$$Z = \epsilon^{1/2}(\zeta - c_o\tau) ; \tag{8}$$

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is also introduced. Here c_o is undetermined.

Introducing (5)-(8) into (3) and (4) gives at $O(\varepsilon)$

$$c_o B_{1Z} = Q_{1Z}$$

and

$$Q_1 = 2B_1 \tag{9}$$

thus

$$c_o = 2 \,, \tag{10}$$

where the letter subscript denotes differentiation. At $O(\varepsilon^2)$, we have from (3) and (4), respectively

$$-B_{1\sigma} + c_o B_{2Z} = Q_{2Z}$$

and

$$Q_2 = B_1^2 + 2B_2 + Q_{1ZZ} \,.$$

Eliminating Q_2 from these equations and using (9) and (10) results in the KdV equation

$$B_{1\sigma} + 2B_1 B_{1Z} + 2B_{1ZZZ} = 0 \,. \tag{11}$$

Rescaling (11) with

$$\eta = 2^{2/3} B_1$$

$$\theta = 2^{-1/3} Z$$

gives the canonical form of the KdV equation

$$\eta_\sigma + \eta\eta_\theta + \eta_{\theta\theta\theta} = 0.$$

Discussion

This same expansion method can be applied to the more general magma flow equations of both Scott and Stevenson [1984, equations (3) and (4)], and Richter and McKenzie [1984, equations (11)-(14), $\phi_o \ll 1$)]. This also results in the KdV equation for the $O(\varepsilon)$ departures from the basic state.

The derivation of the KdV equation described above does not prove that the full equations [(3) and (4)] possess all the required properties for exact soliton behavior. In fact, Barcilon and Richter (1986) find numerically that their one-dimensional magma migration equations do not admit perfect solitary wave interactions. A very weak dispersive tail is generated upon collision. Scott and Stevenson (1984) and Whitehead (in preparation) also observed slight deviations from perfect collisions. However, the derivation does show that the magma migration equations fall into a class of KdV-like equations and soliton behavior might be expected under certain conditions (e.g., $O(\varepsilon)$ departures from the mean state).

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