A boundary layer flow with multiple equilibria

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Convection is driven by both a solute and heat imposed at one side of a vertical slot. Density distribution is calculated using boundary layer theory and flow is determined using a simple viscous flow law. More than one steady flow exists for some ranges of the governing parameters. The solution relates double diffusion and multiple equilibrium circulation concepts. [DOI: 10.1063/1.1480002]

The birth of double diffusion is often attributed to the 1956 description of the perpetual oceanic salt fountain.[1] In that note, the reader is asked to consider a submerged tube extending downward a large distance into the deep tropical ocean so that warmer values of temperature and higher values of salinity are found near the top of the tube, and colder fresh values are found at the bottom. If the tube is filled with warm salty surface water, thermal conduction through the tube wall cools the salty water, making it denser overall than the ocean water outside. Thus a body force is created that allows flow down the tube. Conversely, if the tube is filled with cold fresh deep water, thermal conduction will heat the water in the tube and make the warm fresh water in the tube lighter than the ocean water. Thus a body force is created that allows flow up the tube to continue. The salt fountain is regarded as the discovery of flows driven by double diffusion, since heat diffuses through the tube wall much more easily than salt.

In following years, the studies of double diffusion dynamics advanced rapidly. The works resulted, for example, to new sections in books on convection.[2] Salt finger convection cells and layered flows were two of the many features discovered. Their transport rates are quantified, and physical measurements in laboratories, in the ocean, in volcanic systems, and in engineering have provided ample evidence of their effects.

It was also realized soon after the salt fountain paper appeared[3] that the combined heat and evaporation boundary conditions imposed by the atmosphere can conceivably produce multiple states of ocean circulation. Within certain ranges of the forcing parameter, the states are thereby dependent on their initial condition. This could cause the ocean to exhibit catastrophic or discrete jumps as the boundary conditions are slowly varied due to atmospheric changes. The governing equations are beautiful and simple examples of finite amplitude instability (also called catastrophic) transitions. Further development of this idea was slow; only three studies of thermohaline flows with multiple equilibria were conducted before 1985.[4] Since then, a wide range of conceptual box models and numerical computer models produced predictions of their features and the ensuing consequences. The few physical examples with direct observation[5,6] are restricted to highly controlled laboratory studies.

Here we describe a situation that involves double-diffusion that results in two steady flows with multiple equilibria. Only one other analysis exists to our knowledge where double diffusive driving produced multiple equilibria. This concerns a stability problem. Veronis[7,8] determined the infinitesimal and finite amplitude stability of a layer of fluid with a stable salt difference and an unstable temperature difference imposed on the top and bottom boundaries of a horizontal layer of fluid. Multiple states are possible near the critical Rayleigh number. One flow state is either motionless or oscillatory, depending on Prandtl number, with a stabilizing salinity distribution in the interior of the fluid. The other flow state is a more rapid overturning with stronger mixing. Due to the complexity of the equations, solutions were obtained using numerical methods.

The problem studied here differs from the double diffusion stability case because we consider a solution to a double diffusion driven equilibrium flow problem rather than a stability problem. The result is two steady flows with different speeds and direction. The results can be displayed using analytical solutions.

In deference to the oceanic origins, we refer to the two constituents that affect fluid density as heat and salinity, but any pair of solutes would do as well. In that case, salinity refers to the solute with smaller diffusivity. We also refer to the fluid as water, but it could be any fluid that contains two internal diffusing components that alter density. In this problem, water occupies a vertical slot subject to diffusion of both heat and salt in a field of gravity g. The slot is of width L and has vertical extent H. It is open at the top and bottom and is set in a large reservoir of fresh stagnant water with temperature T_0. The fluid in the reservoir is connected between the top and bottom of the slot so that there is no vertical flow in the slot if heat and salinity effects are absent. One wall of the slot is kept at temperature T_0+T^* and has a salinity S^*. The other wall conducts no heat and allows no salt flux. We consider only linear effects of temperature and salinity on the change in density, so that density of the fluid obeys $\rho=\rho_0(1-\alpha(T-T_0)+\beta S)$. For ease of calculation, the heated wall is taken to be a zero stress wall. We develop a model of heat transfer and dynamics for one dimensional

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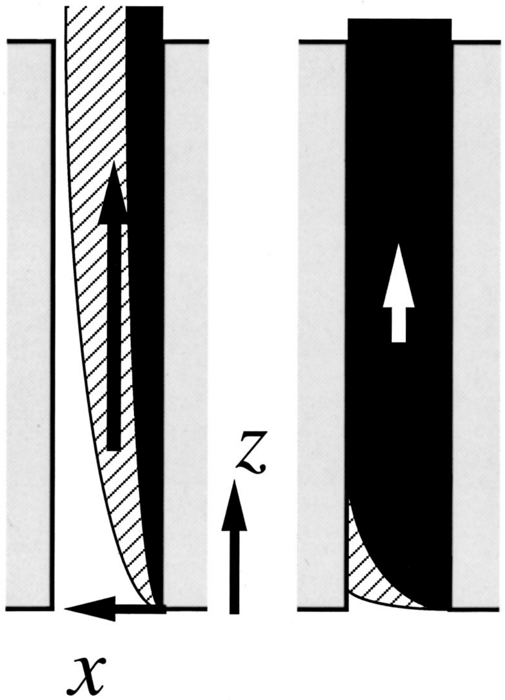

FIG. 1. Left, sketch of temperature (hatched) and salinity (solid) boundary layers for flow fast enough to produce boundary layers that do not extend across the slot. Right, temperature (hatched) and salinity (solid) in developing boundary layers that fill the slot in the limit of slow flow. The arrow indicates the direction and relative speed of fluid flow.

laminar bulk fluid flow with temperature and salinity boundary layers.

First we consider the temperature distribution alone, starting from a state where temperature initially is T_0 everywhere and then temperature of the wall is increased to $T_0 + T^*$ for $t \geqslant 0$. There is initially no fluid velocity in the slot, so a thermal boundary layer will begin to grow following

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}. \tag{1}$$

Here we assume the wall surface is normal to x and the wall extends infinitely in the other two directions. The solution is the well-known complimentary error function[9]

$$T = T_0 + T^* \operatorname{erfc}\left(\frac{x}{\sqrt{2\kappa t}}\right), \tag{2}$$

where x is the cross-slot coordinate with origin at the heated wall, κ is thermal diffusivity, and t is time. With the presence of the gravitational force, but in the absence of salinity effects on buoyancy, a buoyant thermally convective flow will develop and cause vertical flow in the slot. If this vertical flow velocity w becomes sufficiently rapid, the thermal plume will be conveyed upward and out of the slot in a boundary layer as sketched in Fig. 1. Transforming to the upward moving coordinate system of the fluid, a boundary layer develops of the form

$$T = T_0 + T^* \operatorname{erfc}\left(\frac{x\sqrt{w}}{\sqrt{2\kappa z}}\right), \tag{3}$$

where z is the vertical coordinate with origin at the bottom of the slot for upward flow ($T^* > 0$) and with the origin at the top of the slot otherwise. The total upward thermal buoyancy force in this fast limit F_{Tf} (subscript T for thermal and f for fast) per unit of depth can be approximated as

$$F_{Tf} = \rho_0 g \alpha T^* \sqrt{\frac{8\kappa H}{27|w|}}. \tag{4}$$

The constant $\sqrt{8/27}$ can be attributed to the structure of the temperature profile in the boundary layer. The reason for this precise value will be explained soon. This force is balanced by shear stress of size $\mu w/L$ per unit depth at the unheated wall where velocity is set to zero. The vertical velocity at the heated wall produced by this balance is

$$w = \left\{ \frac{8\kappa H}{27} \left(\frac{g\alpha T^* L}{\nu} \right)^2 \right\}^{1/3}, \tag{5}$$

where $\nu = \mu/\rho_0$ is kinematic viscosity. Using this, the condition that the boundary layer thickness $\sqrt{2\kappa H/w}$ is smaller than the slot width is that the Rayleigh number obeys

$$R_a = \frac{g\alpha T^* L^4}{\kappa \nu H} > 3^{3/2}. \tag{6}$$

For smaller values of Rayleigh number, the thermal boundary layer is thicker than the slot in some portions of the slot. We assume that there is a boundary layer in part of the slot and that the rest of the slot has fluid of temperature T^*. The thermal buoyancy force per unit depth in this slow limit is then

$$F_{Ts} = \rho_0 g \alpha T^* L \left\{ 1 - \frac{L^2|w|}{2\kappa H} \right\} \tag{7}$$

(subscript s signifies slow). The right-hand term in the brackets is due to the region with a developing boundary layer. The actual value of the coefficient 1/2 is somewhat arbitrary because of unknown effects of the actual density distribution.

Equation (4) equals Eq. (7) at the critical velocity

$$w_{Tc} = \frac{2\kappa H}{3L^2}, \tag{8}$$

and so we regard Eq. (4) to be valid for $w > w_{Tc}$ and Eq. (7) to be valid otherwise. The constant 8/27 in Eq. (4) was selected to bring it to agreement with Eq. (7) for $w = w_{Tc}$.

The same analysis is easily performed for salinity which has diffusivity D in water. In that case, the buoyancy force can be found from salinity by the substitution of $-\beta S^*$ for αT^* and D for κ in Eqs. (4), (7), and (8). The results are

$$F_{Sf} = -\rho_0 g \beta S^* \sqrt{\frac{8DH}{27|w|}}, \tag{9}$$

$$F_{Ss} = -\rho_0 g \beta S^* L \left\{ 1 - \frac{L^2|w|}{2DH} \right\}, \tag{10}$$

and

$$w_{Sc} = \frac{2DH}{3L^2}. \tag{11}$$

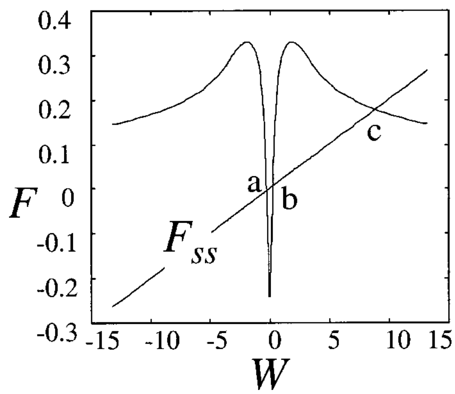

FIG. 2. Buoyancy force as a function of velocity according to Eqs. (12)–(14) in their various limits. Transition between the limits is at $W=0.33$ and 3.33. Also shown is a flow law according to Eq. (15) for $\hat{T}/R_a\tau=0.001$. The line intersects the curves at three points.

Superimposing buoyancy forces of temperature and salinity by using Eqs. (4), (7), (9), and (10) within their respective limits, the buoyancy force as a function of velocity follows a curve that can change sign for certain parameters. We nondimensionalize using $\rho_0 g\beta S^*L$ for the force scale, and $2DH/L^2$ for velocity scale. The body forces per unit depth in the three appropriate limits for a saltwater system (so $\tau<1$) are

$$F=\hat{T}-1+|W|(1-\tau\hat{T}) \quad \text{for } |W|<\tfrac{1}{3}, \tag{12}$$

$$F=\hat{T}(1-\tau|W|)-\sqrt{\frac{4}{27|W|}} \quad \text{for } \frac{1}{3}<|W|<\frac{1}{3\tau}, \tag{13}$$

$$F=\sqrt{\frac{4}{27|W|}}\left(\hat{T}\sqrt{\frac{1}{\tau}}-1\right) \quad \text{for } \frac{1}{3\tau}<|W|, \tag{14}$$

where $\hat{T}=\alpha T^*/\beta S^*$ and $\tau=D/\kappa$. Curves of these results are shown in Fig. 2 for $\tau=0.1$ and $\hat{T}=0.75$.

Let us assume that there is no acceleration so that the total buoyancy force equals shear stress at the wall. Let us also take that stress has magnitude $\mu w/L$ per unit depth, steady flow is determined by the intersection of the curves expressed by Eqs. (12)–(14) and a straight line given by the dimensionless force F_{ss} of shear stress that obeys

$$F_{ss}=\frac{2\hat{T}}{R_a\tau}W. \tag{15}$$

This is also plotted in Fig. 2. The dimensionless number $\hat{T}/R_a\tau$ is a combination of three dimensionless numbers given above. We note that for certain values of $\hat{T}/R_a\tau$, the straight line intersects the curves in more than one location. There is a second condition also required for multiple intersections. It is simple to see that for Eqs. (12)–(14) to have two signs of force the condition $\hat{T}<1$ is required.

The curve is extremely similar to the curve in Fig. 6 by Stommel,[3] and most of the conclusions given in that study apply to this model too. This model has the following virtue: It possesses a very simple internal structure that leads to the multiple equilibria. As in the other model, it is easy to determine the stability of the three points visually. If speed (velocity magnitude) is decreased slightly for point *a*, the buoyancy force is greater but the flow stress is smaller. Thus the fluid will speed up and bring the value of speed back to point *a*. If speed is increased, the fluid will decelerate and bring speed back to point *a*. Thus the steady flow at that point is stable to small perturbations. For point *b* the opposite is true. If speed is decreased slightly, the buoyancy force decreases more than the stress. Thus the fluid in the slot slows down and speed is brought further away from point *b*. An increase in speed results in an acceleration and again speed is brought further away from point *b*. Thus point *b* is an unstable point of fixed flow. And finally, point *c* is a stable point like point *a*.

In summary, the behavior of boundary layer flows for two substances with different diffusivities in confined gaps has a multiple equilibrium nature. A curve that is very similar to the original curve by Stommel is derived using simple boundary layer theory. Although such boundary layers are not likely in large-scale physical oceanography, they may be encountered in other branches of earth science and engineering. Thus this simple problem may be helpful in understanding flows in a number of binary fluids as found in hydrothermal systems, in magmas, in ice dynamics, or in alloys during solidification or melting,

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