J. A. Whitehead, Jr.

Cellular Convection

Experiments in which disordered turbulence gradually emerges from an ordered pattern offer hope for understanding turbulent flows

Humankind lives in a fluid environment; his two most immediate needs, air and water, are fluids. Among the myriad processes involved as a newborn babe emerges from the womb, one of the most vital—and most symbolic to humans—is the first breath, which introduces gaseous oxygen to the living being. He will meet new fluid processes in a variety of encounters, from raindrops in a puddle to the magnificent majesty of a towering cumulus. Dripping faucets will result in long, dull, sleepless nights, while bubbly wine will tickle his nose. A documentation of man's conquest of fluid processes includes a relatively comprehensive history of man himself.

In the same light, a scientist involved in the study of fluid dynamics can often find application of his discipline in a variety of applied subjects, ranging from macroscopic astrophysics to microscopic biophysics. The subject of this article, cellular convection, shown in Figure 1, is no exception.

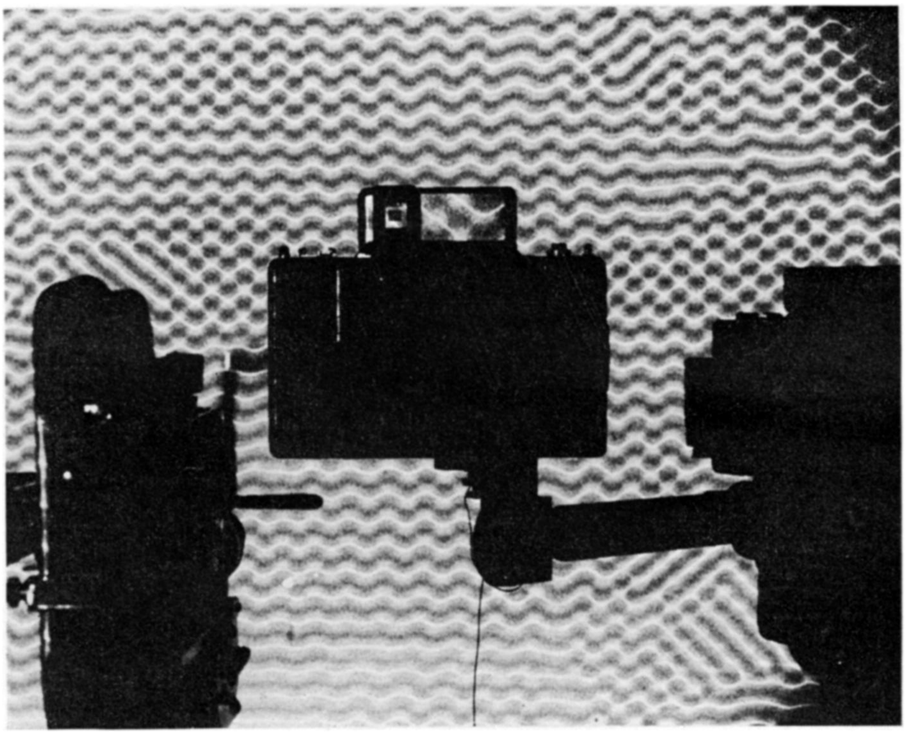

Figure 1. Movie and still cameras recording the evolution of an instability to cellular convection rolls.

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Cellular convection exists in its simplest form when a thin horizontal layer of fluid is heated from below so that the warm fluid near the bottom tends to float up buoyantly and displace the denser fluid above. One finds that for low heating rates, a disturbance cannot sustain itself against the combined stabilizing influences of thermal conduction, which smooths out the thermal disturbance, and viscosity, which damps motion. For greater heating, experiments demonstrate a collection of well-ordered roll-like cells whose wavelength must lie within two very close limits, as shown in Figure 2.

Fluids, in general, are capable of very complex flows, as the previous examples show, and a complete analytic solution of the "laws of motion" would be virtually impossible. In addition, the behavior of fluid in many cases is nonlinear, meaning that one cannot necessarily utilize the principle of superposition to obtain a number of relatively simple solutions which can then be added together. Fortunately, one can group together problems that share similar dynamics. In this vein we say that cellular convection is one of the simplest examples of *interchange instabilities*, in which an exchange of neighboring blobs of fluid leads to internal forces that tend to continue the motion. As such it often serves as an archetype for more complicated flows in nature.

Three examples come to mind. The first is a model of motions inside the viscous mantle of our planet. It has been suggested that convective cells are rising from the hot interior of the earth, which might generate the continental drift motions seen on the crust of the earth. The model is still hazy since viscosity is believed to vary greatly in the mantle, the crust of the earth is believed to behave as a semirigid plate, and a careful account of heat production inside the earth is not possible. It falls to experimenters and theoreticians to assess the influences of these factors on the simple cellular model.

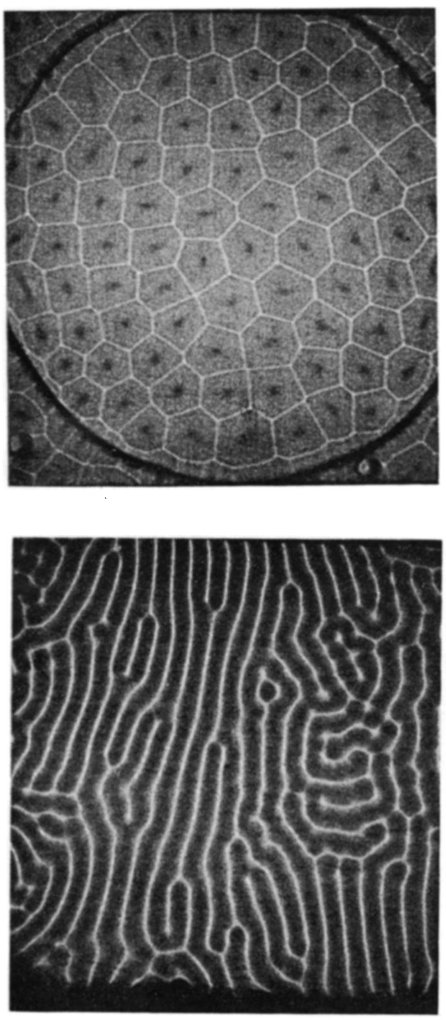

Figure 2. Shadowgraph pictures of convection as seen from above. The white lines correspond to cool fluid plunging into the interior from the top surface, and the dark lines to hot fluid coming up from the bottom. The top picture shows cells somewhat similar to the ones seen by Bénard in the late nineteenth century; the top surface was exposed to air. The bottom surface shows roll-like structures which occur when the fluid is bounded above and below by rigid non-slip boundaries.

A second example occurs in the top layers of the ocean and in the mixed region of our atmosphere. Solar energy, after heating the earth and oceans, is transported upward by a convective process, sometimes generating the strikingly well-ordered cumulus patterns seen in our atmosphere. On the average, such motions must occur on an earthlike planet because moist air is not transparent enough to re-radiate outward as much solar energy as falls on the earth. Again the suitability of simple convection as a prototype of these processes must be assessed.

The third example of cellular convection is found in a layer of our sun that is relatively opaque to radiant energy coming from the interior. Astronomers who have photographed the cellular-type layer of the sun have coined the word "granulation" to describe this effect; it is believed to be an important process in many types of stars. There is presently an active effort to understand the granulation process in light of what we know about cellular convection.

In addition to such interesting applications there is a second, more fundamental, reason for studying these flows, namely that they demonstrate how turbulence emerges in a fluid flow. Turbulence consists of rapidly fluctuating flow patterns which do not bear any apparent resemblance to the boundary conditions of the flow. In most experiments the transition to very complicated turbulence occurs suddenly, making it difficult to see any particular structure in the flow. However, cellular convection is one of the most ideal methods of studying turbulence, because the turbulent patterns emerge more gradually and one can study the physical processes which determine how the fluid adopts each new pattern.

One of the more common examples of cellular convection led to an early observation of the effect in 1882, when James Thompson (*1*) was hiking through the countryside with the Belfast Outing Club. He and a group of friends paused to rest near a building when their attention was attracted to a pail of hot soapy water in which sheets of relatively clear surface fluid were plunging downward into the dirty interior due to cooling from above. The sheets met to form the vertices of polygons that presented a kaleidoscope of changing shapes and patterns as the vertical sheets changed their size and orientation. The physical process which would produce these tesselated polygons intrigued him to the point that he discussed their origin before the Royal Society of Glasgow.

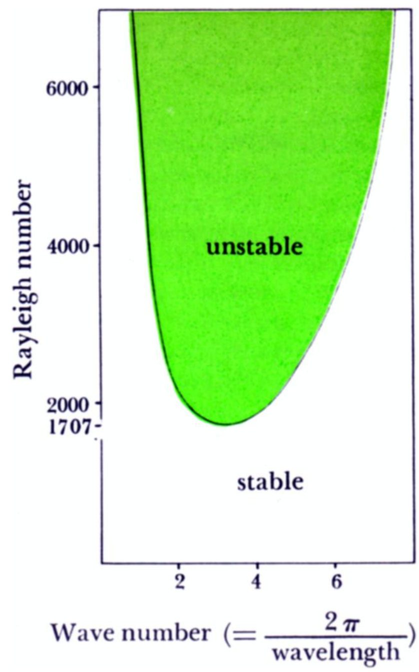

Figure 3. Stability diagram of disturbances in a motionless fluid heated from below. One can infer from this that a Rayleigh number in excess of 1707 is sufficient for motion to occur.

More intriguing yet were the extremely regular patterns generated in the laboratory by Henri Bénard, a Frenchman who did a series of very carefully conducted experiments in which a horizontal layer of oil was heated from below. As reported in 1901 (*2*), he observed two principal features of this motion: first, he noted that with very low heating the fluid remained motionless; second, he observed that when motion commenced it took place as a lattice of very regular polygons. The polygons then slowly shifted until eventually they arranged themselves as a perfectly regular honeycomb structure of hexagons. This pattern intrigued him, and apparently others, principally because it was hoped that in certain circumstances as fluid fell away from an equilibrium value the first types of motion to be expected would have the regularity that these polygons exhibited. Unfortunately, there are not many examples of fluid flows where this happens—in most situations the fluid goes unstable cataclysmically, resulting in very complicated and disorganized turbulence.

Linear stability

In 1916 Lord Rayleigh (*3*) analyzed the equations of a small disturbance in a motionless fluid heated from below, and he was led to functions expressing the growing disturbance. The equations resembled those of free vibrations in a stretched membrane; in fact, the problem reduced to an eigenvalue problem similar to those Rayleigh discusses at length in his *Theory of Sound*. He was therefore well equipped to handle this aspect of cellular convection, and his two findings—that an infinitesimal disturbance will grow if the temperature difference of the layer exceeds a certain critical value, and that the most unstable disturbance has a certain unique wavelength (see Fig. 3)—have been confirmed, experimentally and theoretically.

Unfortunately, in the years closely following Rayleigh's analysis, no new physical or theoretical concepts of the selection process were introduced. Possibly the emergence of exciting new fields, such as quantum mechanics and relativity, seduced physicists away from this more classical study. Apparently most people were satisfied that the stability problem of a motionless fluid was at the limit of tractability, and so the efforts that were made improved the precision and scope of the linearized theoretical equations used by Rayleigh.

In addition, a number of experiments were conducted to observe the flow more fully. It was found that the hexagonal structure so nicely observed by Bénard was strongly a function of the surface tension of the free surface on top of the fluid, and recent theories have shown that Bénard's original cells were surface tension driven. In most instances where the surface tension force was lacking, a row of two-dimensional roll structures would result; hexagons are a result of a more complicated dynamic process, to be discussed later in this article. The rolls still had the same intriguing spatial periodicity as the hexagons, with a slight exception that the rolls were not always perfectly regular, but tended to curve. An exposition of such work, studying in detail this and many other problems where a small arbitrary disturbance begins to grow about some marginally stable state, by S. Chandrasekhar (*4*), was published in 1961.

To proceed further entailed severe theoretical difficulties as the equations became nonlinear, and most standard methods of calculus became useless. How could one proceed to incorporate the nonlinear terms in the equations and get a final steady solution valid above the critical temperature difference? In this case the Rayleigh analysis indicated that a variety of wavelengths could grow, and it was unclear why only one wavelength would eventually grow to finite size, as experiments showed.

The nonlinear terms were of the form $\mathbf{U}\cdot\nabla\, T$, where $\mathbf{U}$ is the unknown velocity of the fluid and T is the unknown temperature. Rayleigh solved his infinitesimal stability equations by saying that the largest component of this term, for small velocities, was $W\beta$ where β is the (known) temperature gradient of the undisturbed fluid and W is the vertical velocity.

A scheme to generate a nonlinear solution was developed by W. V. R. Malkus and G. Veronis (*5*) in the United States and independently by L. Gor'kov (*6*) in the USSR. The solution procedure, a refinement of earlier works by J. T. Stuart (7), employed Rayleigh's equations as a first approximation for the flow, and then solved for a correction due to the nonlinear terms. The solution was valid because the temperature gradient of the fluid would be larger than any temperature perturbation of the fluid if the system was sufficiently close to the critical temperature difference above which motion began.

In other words, the velocity and temperature disturbances that grew to finite size could be made to be as small as desired by getting arbitrarily close to the critical temperature difference, and in fact could be made smaller than β, the temperature gradient produced by heating from below. Formally, this could be envisioned as setting the dimensionless temperature difference (called the Rayleigh number) as $Ra = Ra_c(1 + \gamma)$ where γ is small, and then expanding velocity and temperature

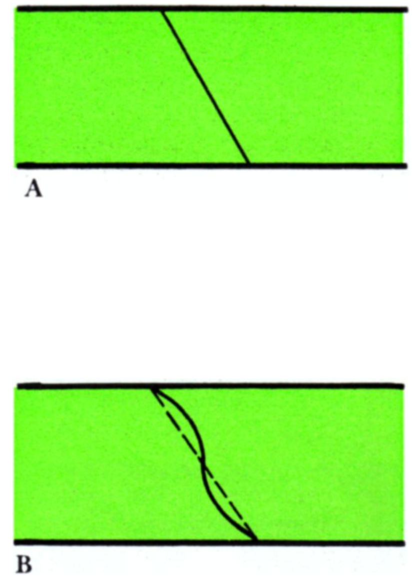

Figure 4. Mean temperature gradient of A, motionless fluid, and B, fluid in motion. Note that convection is accompanied by a decrease of the average temperature gradient, which is equivalent to a decrease in the average gravitational potential energy of the fluid.

as an asymptotic series of the small parameter ϵ

$$W = \epsilon W_1 + \epsilon^2 W_2 + O(\epsilon^3)$$

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3)$$

When these solutions are put into the differential equation, certain terms must be set equal to zero to avoid a mathematical resonance which would generate an infinite term. This results in a condition relating the amplitude of the disturbance ϵ to the excess Rayleigh number γ such that

$$\epsilon = \sqrt{3\pi^2\gamma}$$

The resulting analysis correctly predicted the magnitude of fluid velocities and the approximate excess heat transported by the motion.

A principal result was that the most important nonlinear effect of the finite, but small, motion was to pump heat in such a way that the mean temperature gradient was decreased. In this way the fluid was made stable to all other disturbances and they would not grow to a finite size (see Fig. 4).

At a given point in the analysis it was found that an assumption would have to be made as to the type of flow that existed—whether it was hexagons, squares, rolls, or something else. Calculations then had to be completed as to whether this type of flow was stable and could exist. In other words, this procedure required an a priori assumption based upon the disturbance wavelength and the form of the convective pattern that would exist. It was felt that rolls or rectangular cells or square cells of a variety of sizes could exist in a stable manner. The question then remained as to what physical processes determine the pattern that is viewed in a given experiment. This problem has been the topic of active interest for the past ten years, and it appears now that many of the principal processes under study have been analyzed.

Shortly after Malkus and Veronis developed their mathematical scheme, there was similar progress in Germany by A. Schlüter, D. Lortz, and F. Busse (*8*), who subsequently went somewhat further and took the resulting motions and mathematically subjected them to infinitesimal perturbations to find whether these motions were stable. It was found that neither three-dimensional hexagons nor squares nor rectangles were stable because perturbations grew in their presence, and also there was a bandwidth of wavelengths for two-dimensional rolls that could exist in a stable manner. This bandwidth was considerably smaller than the bandwidth of the solutions which would grow according to the linearized infinitesimal Rayleigh theory. If a roll was outside this bandwidth because of some initial conditions in the experiment, perturbations of a very different character would begin to grow.

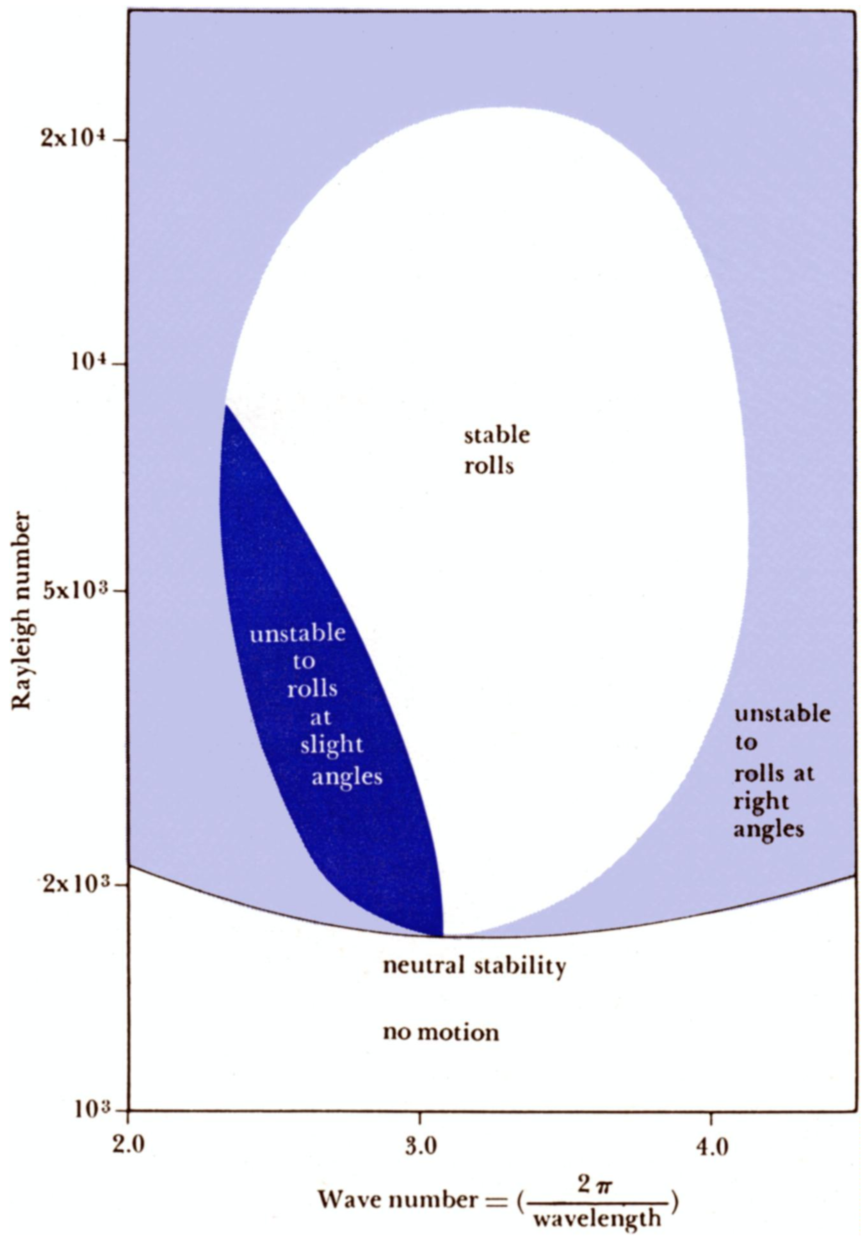

Figure 5. Stability diagram of rolls, after Busse (*9*).

For instance, if the rolls were too wide they would be unstable to other rolls of smaller width and at a slight angle to the original ones; if the rolls were too narrow, they would be unstable to larger rolls at right angles to the original ones. More extensive numerical calculations on the stability of rolls by Busse (*9*) have resulted in the stability map shown in Figure 5. There is a region on the Rayleigh number-wave number plane for stable rolls. The question of what size roll would exist in an actual experiment, however, was not precisely answered because there was still a bandwidth of possible roll sizes left in a stable state; the mechanism by which the nonlinear selection processes selected among all stable candidates to make the strikingly periodic rolls observed was not completely isolated.

L. Segel (*10*) attacked the problem of what would happen if two rolls existed simultaneously. It was found that if the amplitudes were small when they started, both would grow, but as they got larger they would interact in such a way that one would win out over the other and end up by ultimately absorbing all available energy. This was also found to be true for a larger number of discrete rolls. Although there was some question as to whether neglected terms in the equation would produce additional rolls which might alter the results, Segel's equations showed that many sizes and orientations of rolls would grow initially but a well-defined selection process determined which rolls would finally persist. That the final state consists of only one set of well-determined rolls has been observed experimentally by Silveston (*12*) in Germany and Koschmieder (*13*) in the United States, and has even emerged in numerical calculations by Deardorff (*14*) for two-dimensional regions wide enough to admit one couplet of cells, and by Foster (*15*) for wider regions.

Stimulated by these results, M. Chen and the author (*16*) developed a technique for looking at rolls of arbitrary initial size. Motionless fluid below the critical Rayleigh number (Rayleigh number is proportional to temperature difference—when a Rayleigh number exceeds 1707 motionless fluid is unstable) was preheated with a spatially periodic heat pattern. The fluid was then raised above the

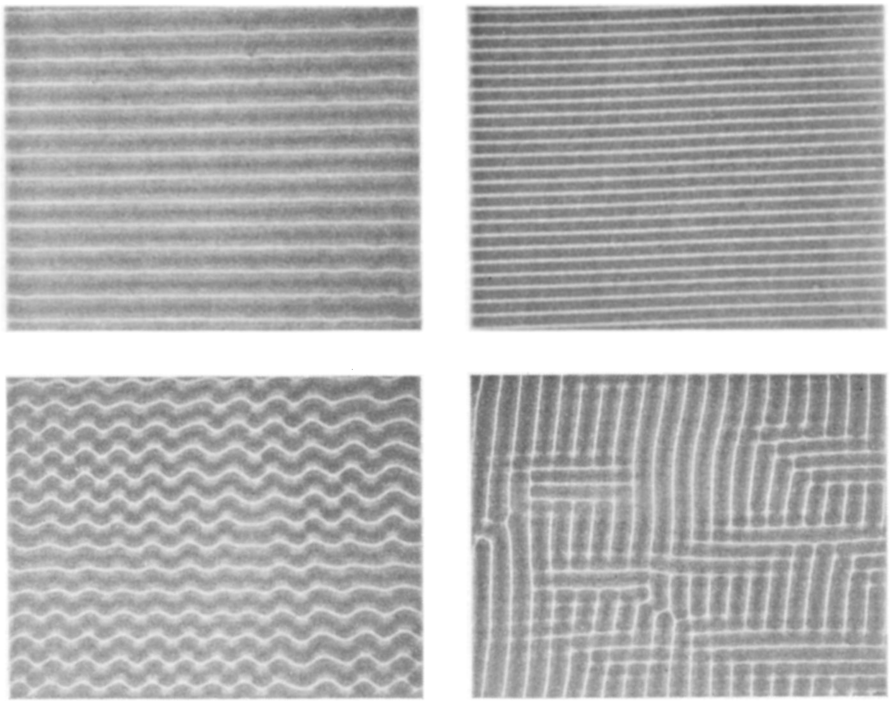

Figure 6. Laboratory photographs of rolls becoming unstable. *Left*, the rolls are large enough to be unstable to rolls at slight angles. *Right*, the rolls are small enough to be unstable to rolls at right angles.

critical Rayleigh number to the desired steady state, the preheating was stopped, and, since the preheated wavelength had such an amplitude advantage over random noise wavelengths, it would persist and would be observed as the initial roll size in the experiment. It was then possible to see what would happen to the rolls at subsequent times—whether they would be unstable or stable, and what the end motion would look like.

The most striking aspect was the regularity and periodicity of the new rolls that would come in if the original rolls were too large or too small. When the rolls were too small they indeed did become unstable to rolls at right angles, as predicted by Schlüter, Lortz, and Busse and as illustrated by recent pictures, taken jointly with F. Busse, in Figure 6. The original rolls can be seen, and a later picture shows the rolls coming in at right angles and completely obliterating the original rolls. When the rolls are too large, the instability to smaller rolls at slight angles is evidenced by the fact that the rolls begin to zigzag. The pictures show that the zigzags grow until they reach approximately 45 degrees, and then they break down.

Recent refinements have continued to broaden the scope and power of the theoretical techniques. A. Newell and the author (*17*) and, independently, Segal (*18*) have used a theoretical technique basically similar to that used by Malkus and Veronis ten years earlier, but with the refinement that the form of convection does not have to be specified a priori. The results were put in the form of a relatively simple solvability equation showing that a large class of flows, most having a roll-like nature, are possible as steady solutions. In addition, initial conditions strongly influence the exact size and configuration of the rolls that result, and boundaries can also have an effect. This solvability equation exhibits the same instabilities found originally by Schlüter, Lortz, and Busse. The most recent analysis by A. Newell, C. Lange, and P. Aucoin (*19*) of the initial value problem with a continuous random spectrum shows that the system is driven to a discrete spectral spike—that is, a rather random initial condition is definitely driven toward a roll solution.

The mathematical technique and the associated experiments have served to clarify the processes involved in cellular convection. First, Rayleigh demonstrated that certain modes will liberate enough potential energy to overcome friction and thermal conductivity if the temperature gradient is sufficiently great. Malkus and Veronis showed that these modes will grow until nonlinear processes balance potential energy liberated by the cellular flow, the most important effect being that convection will alter the mean temperature profile and arrest further growth of the disturbance. Lastly, Schlüter et al., Segal, and now Newell et al. have shown the interplay of various modes and their stabilizing influence upon each other. These three factors together cause a disturbance to grow and create its own stable equilibrium and so to generate the strikingly periodic and stable motion of cellular convection.

As the reader will remember, Bénard observed a hexagonal planform of cells rather than rolls. Analysis indicated that the processes generating such hexagons were a shade more subtle than those associated with rolls. Briefly, hexagons are an asymmetric flow above and below the horizontal midplane of the fluid; hot, upwelling fluid either ascends as a column in the center of the hexagons, or it ascends up the sides of the hexagons and descends as a cool column. It was found that the unmoving fluid must have some associated asymmetry above and below the horizontal midplane arising either from viscosity variation, surface tension, or a nonuniform density gradient. The role of surface tension in Bénard's original experiments was noted by Palm (*20*), while Busse (*21*) explored the role of viscosity and all other material property variations. Briefly, the hexagonal solution was found to be associated with a finite amplitude instability, which could be expressed by the following equation for r, the amplitude of hexagons:

$$\frac{\partial r}{\partial t} = \left(\frac{Ra - Ra_c}{Ra_c}\right) r + Cr^2 - r^3$$

Here Ra is the Rayleigh number while Ra_c is the critical Rayleigh number, i.e. the minimum, in Figure 3. The constant C is determined by the asymmetry of the hexagons with respect to that of the density profile of the unmoving fluid. It is evident that if $(Ra - Ra_c)/Ra_c$ is negative, the plus C solution admits growing r for r sufficiently large. The constant C is generated because a hexagon is actually three sets of rolls rotated 120° with respect to each other; these three rolls interact in a nonlinear *triad resonance* to generate the finite r. The triad resonance is one of the simplest and probably most common nonlinear interactions in multidimensional mathematical physics, and as such plays an important role in many modern theories.

In connection with hexagonal flows, the triad resonance can be briefly viewed as follows: the heat pumped by convective motion is proportional to the differential operation $(\mathbf{U}\cdot\nabla)T$, where $\mathbf{U}$ is a velocity vector and T is the temperature of the cell. One can represent three rolls as $e^{i(\mathbf{k}\cdot\mathbf{x})}$, where $\mathbf{k}$ is a two-vector in the horizontal plane and where $\mathbf{x}$ is the horizontal coordinate such that $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ and $|\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}_3|$. This can only be done if the two-dimensional $\mathbf{k}$ vectors form the three sides of an equilateral triangle, i.e. if the three rolls add up to form a hexagonal pattern. Now these rolls will form a closed solution under the $(\mathbf{U}\cdot\nabla)\ T$ operation; that is, if we take $\mathbf{U} = e^{i(\mathbf{k}_1\cdot\mathbf{x})}$, $T = e^{i(\mathbf{k}_2\cdot\mathbf{x})}$, and multiply, we generate a solution proportional to $e^{-i(\mathbf{k}_3\cdot\mathbf{x})}$. Likewise, no new harmonics are generated for all other permutations of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. Physically, this means that the $\mathbf{k}_1$ and $\mathbf{k}_2$ rolls pump heat into the $\mathbf{k}_3$ roll and keep it going; likewise $\mathbf{k}_1$ and $\mathbf{k}_3$ pump heat to $\mathbf{k}_2$, while $\mathbf{k}_2$ and $\mathbf{k}_3$ pump heat into $\mathbf{k}_1$. This generates a self-perpetuating flow, which manifests itself as a finite-amplitude subcritical instability.

If one solves for the heat transfer of hexagons as a function of the temperature difference of the unstable layer, a curve is generated as in Figure 7, the dip in the temperature difference being a manifestation of the finite amplitude behavior of the hexagons driven by the triad resonance.

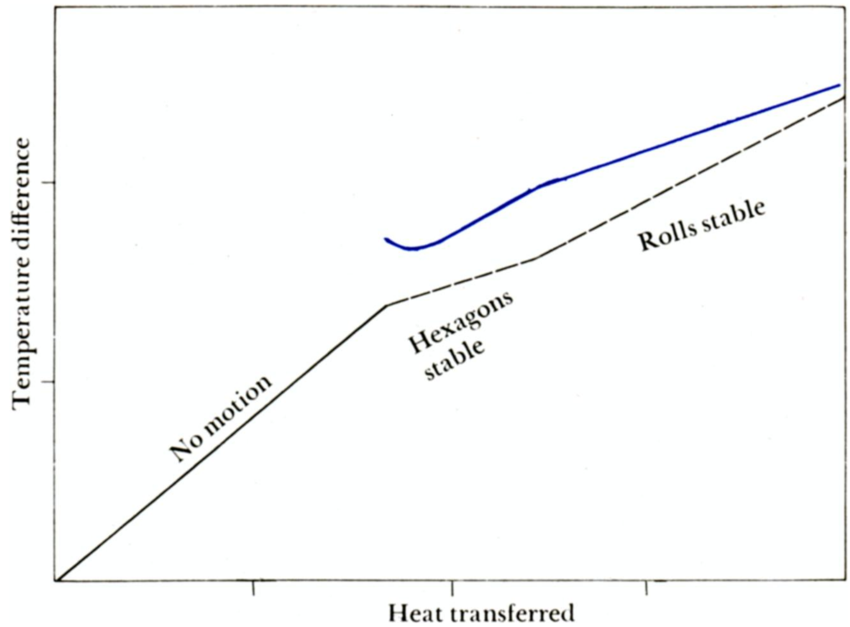

Figure 7. Heat transfer curve of hexagonal convection compared to the heat transfer of rolls. The physically observed pattern will follow the trajectory which maximizes heat flux, and thus the heat flux curve has a dip accompanying the hexagons and then a discrete change in angle where hexagons break down into rolls. Theory indicates a small overlap between the regimes where hexagons and rolls are stable.

Large Rayleigh numbers

Cellular rolls are therefore a simple, yet realizable, example of infinitesimal instability while hexagonal cells are a simple example of finite-amplitude instability. These solutions are generally valid if $(Ra - Ra_c)/Ra_c$ is small—that is, the procedure generates a valid solution for experiments slightly above the critical Rayleigh number. There is an entire spectrum of greater Rayleigh numbers in which other interesting processes occur and which are well served by the interesting analyses done close to the critical Rayleigh number.

The evolution of ideas on large Rayleigh number flows began in the middle thirties when a number of engineering problems stimulated interest in the heat transferred by the convective motion, and a variety of experiments were performed that showed the change in the transfer of heat as one went to higher and higher Rayleigh numbers. The parameter that measures the heat transfer is generally called a Nusselt number, and basically it is a measure of the balance between convection and conduction of a fluid. When the fluid is motionless the Nusselt number is one, and when the fluid is highly convecting the number is large compared to one.

In 1954 the heat transfer observations were greatly refined by W. V. R. Malkus (*22*). In a paper presented in the Proceedings of the Royal Society he reported that, as the Rayleigh number was progressively increased, the slope of the Nusselt number curve changed at discrete points (Fig 8). In a subsequent paper (*23*) he suggested that at each change in the slope the fluid was going to a greater and greater degree of chaos or disorder. These findings were directed in a very strong sense toward the hope Rayleigh expressed in his original paper—that fluids might fall from equilibrium in an ordered way. It appeared that the cellular convection undergoes a series of discrete transitions to more complicated flow as the Rayleigh number is increased. After several years, the experiments showing the transitions were done by others, for instance Willis and Deardorff (*24*) at Boulder, Colorado, and although some of the points at which the slopes change have been debated, the basic findings were confirmed.

These experimental studies have stimulated theoretical models of the high Rayleigh number convection. In all cases except one, the models have relied upon one or more assumptions. Malkus presented a theory concurrently with his experimental report of slope transitions which in essence envisioned each eigenmode of the Rayleigh stability equation as supplying a separate transfer of heat. This led to the asymptotic behavior in the limit of large Ra of $Nu \approx Ra^{1/3}$. Recently, direct numerical solutions of the Malkus equation by Catton (*25*) have led to very good agreement with experiments. Similar models by Spiegel (*26*) and Herring (*27*) have led to the same result.

A model of convection by Robinson (*28*) which examines the dynamics of a roll in the limit of large Ra also leads to an $Ra^{1/3}$ law. The question of whether this roll is stable has not been completely analyzed, although there is experimental and theoretical evidence that it is not. Kraichnan (*30*), using mixing length theory, and

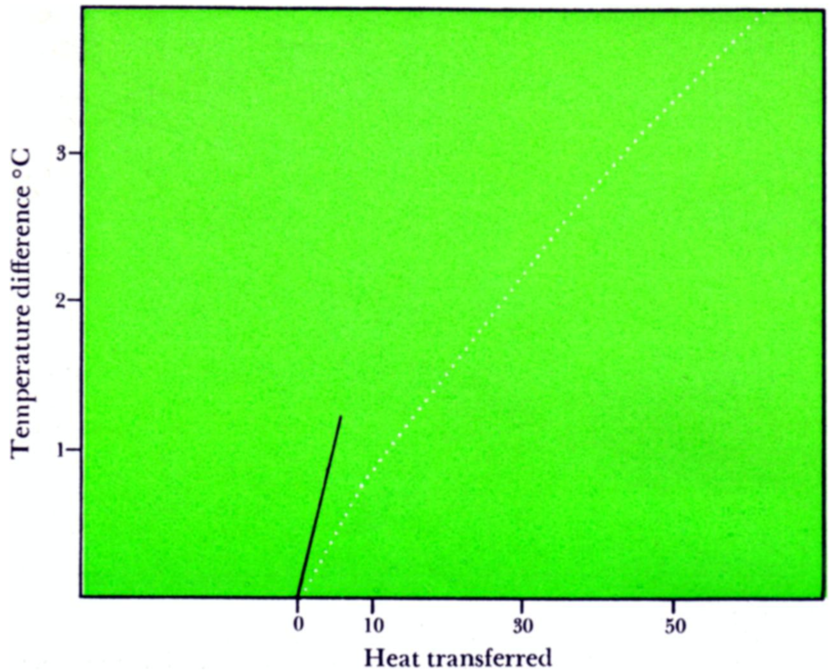

Figure 8. One curve of Malkus' data of heat transfer of cellular flow showing the change in slope of heat transfer.

Figure 9. Pictures of biharmonic flow in which two sets of rolls co-exist at right angles. Rayleigh number from top to bottom is 10,000, 25,000, and 35,000.

Howard (*37*), using estimates of the time it takes blobs to break away buoyantly from the boundary, have also arrived at an $Ra^{1/3}$ law. Kraichnan, however, predicts that shear turbulence will ultimately generate an $Ra^{1/2}$ law for very large Ra. Many of the models differ in details of the fluid motion, but it is quite apparent that the various methods of dealing with the "turbulent motion" are useful.

An exciting approach has emerged in the past score of years in which a mathematically rigorous upper bound is calculated for the Nusselt number. First done by Howard (*32*), extension of this idea by Busse (*33*) has led to the relation $Nu \approx Ra^{1/2}$. It is reasonable to expect that further development of this method will result in a new and very powerful tool in analyzing turbulent flows.

Experiments have been conducted to observe the properties of cellular convection at higher Rayleigh numbers. Somerscales (*34*, *35*) has conducted a variety of experiments measuring various features of the flows and temperature fields as predicted by the theories, as have Deardorff and Willis (*36*). They have not found complete agreement with any one theory, but have usually noted some features of each.

More recently the attempt to find the qualitative change in the roll behavior at the first transition reported by Malkus and predicted by the numerical calculations of Busse, shown in Figure 5, has been done by R. Krishnamurti (*37*) at The Florida State University, Tallahassee, and some experiments have been subsequently done by F. Busse and the author (*38*) at the University of California, Los Angeles. Krishnamurti found that indeed the slope did change at a value of $Ra = 12\ Ra_c$, and that the roll pattern broke into some kind of three-dimensional flow. At UCLA photographs of the convective cells show new rolls appearing at right angles to the original rolls and existing in a stable manner concurrently with the original rolls, as shown in Figure 9. We have suggested that the new pattern be called "bimodal flow." This seems to be the next equilibrium state that persists up to some other transition point at a high Rayleigh number. Bimodal flow also commences in hexagons, as shown in Figure 10.

Krishnamurti (*39*) has also seen the next transition; it appears to be the emergence of an oscillating flow which first commences in one or two spots and gradually spreads over more of the area as the Rayleigh number increases. Another transition has been reported by Krishnamurti at a Rayleigh number of 120,000, at which time a second harmonic of the oscillation becomes apparent in addition to the first. Our own experiments show that the oscillations occur in the bimodal flows, where rolls fit together poorly. These flows rapidly become more energetic and important at Rayleigh numbers above 50,000. By inducing flawless bimodal rolls in our experimental apparatus, the transition to oscillating flows can be delayed to Rayleigh numbers in excess of 200,000, above which the pattern begins to oscillate in a spatially homogeneous way.

All of these descriptions are consistent with the ideas originally sketched by Malkus, and one might hope that by studying these transitions it would be possible to begin to find methods of expressing the emergence of more

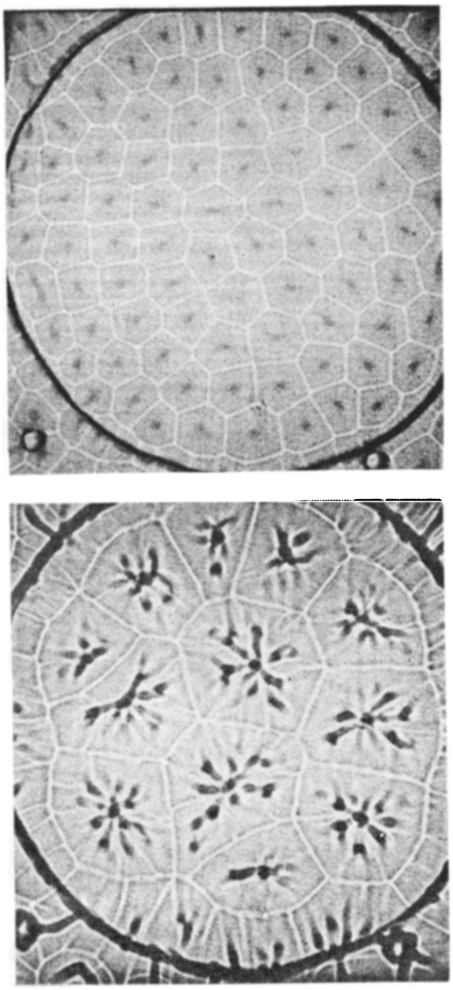

Figure 10. The emergence of biharmonic flow in hexagonal convection. Rayleigh number for the top is 5,000; that of the bottom is 30,000. Depth of the fluid in the larger Rayleigh number experiment is twice that of the smaller.

complicated flows. The years ahead offer excitement as the physics of these higher transitions is uncovered. Particularly intriguing are the powerful new upper bounding techniques, which have already been shown by Busse to possess the same qualitative features of discrete transitions observed experimentally. The true potential of these new techniques has yet to be fully explored.

We note that turbulent flow can be described as non-unique, non-steady, and non-reproducible, and already cellular convection has been found to be both non-unique and non-steady. It is reasonable to expect that the further transitions will contain yet more features of turbulent flows. We close by noting the irony of the fact that this fruitful work was originally stimulated by the beautiful surface tension-driven experiments of Bénard—which involved a different phenomenon!

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