Geostrophic Versus Critical Control in Straits*

LAWRENCE J. PRATT

Department of Physical Oceanography, Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

10 September 1990 and 13 November 1990

1. Introduction

The concept of geostrophic control of a steady flow by a sea strait, first established by Garrett and Toulany (1982), has been the subject of numerous debates and investigations [Garrett 1983; Toulany and Garrett 1984; Garrett and Majaess 1984; Whitehead 1986; Rocha and Clarke 1987; and Wright 1987]. Despite this attention, doubt and confusion continue to exist concerning the basic concept and its formal range of validity. For example, it is not known whether the formulas of rotating hydraulics (e.g., Whitehead et al. 1974 or Gill 1977) reduce to those of geostrophic control in any limit or whether geostrophic control provides bounds on the transports predicted by hydraulic theory.

In this note it is argued that the steady flows described by existing, deductive hydraulic theories are not geostrophically controlled in any limit, nor does the transport relation given by geostrophic control provide any bound on the value predicted by hydraulic theories. It is further argued that geostrophic control occurs when advective effects (which are essential to the behavior of hydraulically driven flow) are overwhelmed by time dependence. This restriction places lower bounds on the values that the characteristic frequency ω of motion can take. At the same time geostrophic control theory assumes time-dependent effects to be *weak* in the sense that $\omega \ll f$, where f is the Coriolis parameter. Thus, time dependence must be weak, but not too weak.

2. Geostrophic control

The fundamental ideas leading to geostrophic control can be described by reference to a strait connecting two semi-infinite basins (Fig. 1). The reader is referred to Toulany and Garrett (1984, hereafter designated TG) for details of the calculation. The basins and the strait have uniform depth H, the water has uniform density, and the elevation of the sea surface is given by ζ (assumed constant in the basin interiors). In the limit of steady flow a mean sea level difference $(\zeta_1 - \zeta_2)$ is assumed to exist between the two basins. Such a flow could be established from a motionless initial state in which a barrier placed at the narrowest width $(x = 0)$ separates the two basins. If $\zeta_1 > \zeta_2$, removal of the barrier leads to flow into basin 2 set up by Kelvin waves propagating away from the barrier. These waves keep the coastline on their right (in the Northern Hemisphere) as shown by the arrows in Fig. 1. If rotation is sufficiently strong each Kelvin wave will be trapped against its respective wall and each will induce only weak motion along the opposite wall. Thus, one might expect the elevations ζ_4 and ζ_5 to remain close to the interior values ζ_1 and ζ_2, since the former do not lie in the path of Kelvin wave propagation. Based on this scenario TG assume

$$\zeta_4 = \zeta_1 \tag{2.1a}$$

$$\zeta_5 = \zeta_2. \tag{2.1b}$$

In addition, the width averaged along-strait velocity $\bar{u}$ is assumed to be geostrophically balanced:

$$\bar{u} = g(\zeta_4 - \zeta_3)/fw = g(\zeta_6 - \zeta_5)/fw \tag{2.2}$$

where g is the gravitational acceleration, f the Coriolis parameter, and w the strait width (assumed constant in TG).

Finally, TG assume a (linear) along-strait momentum balance between friction, local and Coriolis accelerations, and pressure gradient, a statement essentially of the form

$$u_t - fv = -g\zeta_x - \lambda u \tag{2.3}$$

where λ is a friction coefficient and (u, v) are the x- and y-velocity components. (Toulany and Garrett use a width averaged version of 2.3.) In the limit of steady, frictionless motion in a gradually varying strait $(v \ll u)$ the along-strait pressure gradient ζ_x vanishes:

$$\zeta_3 = \zeta_5, \quad \zeta_4 = \zeta_6. \tag{2.4}$$

Combining (2.1), (2.2), and (2.3) results in an expression for the volume transport through the strait. In the

* Contribution No. 7517 from the Woods Hole Oceanographic Institution.

Corresponding author address: Dr. Larry Pratt, Woods Hole Ocean. Institute, Clark-3, Woods Hole, MA 02543.

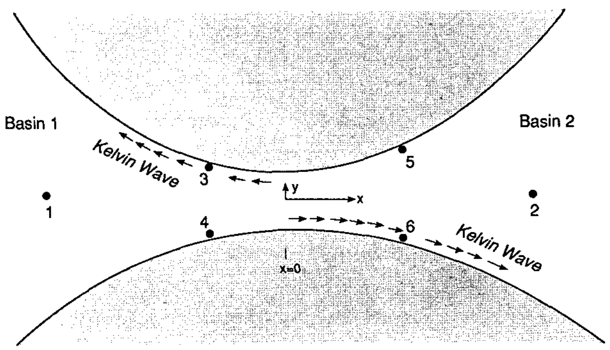

FIG. 1. Sketch of strait connecting two semi-infinite basins.

limit of vanishing friction coefficient and frequency, this expression reduces to

$$Q = Hw\bar{u} = gH\Delta\zeta/f \quad (2.5)$$

where $\Delta\zeta = \zeta_1 - \zeta_2$. The transport is proportional to the drop in sea level between basins, which, in turn, equals the cross-strait sea level difference. Toulany and Garrett argue that, even under more general dynamical circumstances, the cross-strait sea level difference cannot exceed $\zeta_1 - \zeta_2$, so that (2.5) represents an upper bound on the transport. The term geostrophic control is thus used to describe the limit in which (2.5) applies (this upper bound is also the transport in a boundary current set up by a Kelvin wave of amplitude $\Delta\zeta$). Toulany and Garrett compared their solution to a linear analytic solution due to Buchwald and Miles (1974) for oscillatory flow through a narrow gap. In the limit of vanishing frequency the transport is given by (2.5), despite the fact that the assumptions $\zeta_4 = \zeta_1$ and $\zeta_5 = \zeta_2$ are not made.

Rocha and Clarke (1987, hereafter referred to as RC) reconsider the model of TG in the case where the strait and basins have unequal depths. Their calculation involves solutions to the linear Laplace tidal equations

$$u_t - fv = -g\zeta_x \quad (2.6)$$

$$v_t + fu = -g\zeta_y \quad (2.7)$$

$$\eta_t + Hu_x + Hv_y = o \quad (2.8)$$

in a rectangular strait, matched with the basin solutions. In the limit of low frequency ω, the transport through the strait is similar in form to (2.5), but with corrections for the depth discontinuities. This result can be anticipated by reconsidering the Kelvin wave adjustment problem mentioned earlier. When the waves generated by the barrier removal reach the end of the strait, some of the wave energy is carried across the strait along the discontinuities in depth.

A qualitative representation of the streamline patterns for the steady flow found by RC is shown in Fig. 2 for equal basin and strait depths (see their section 3.3). The transport between basins occurs in a boundary layer set up by Kelvin waves propagating away from the gap or strait. As it passes from one basin to the other, the boundary layer switches from one coastline to the other. The crossing region, which contains sharp corners and strong offshore velocities, is set up in part through the action of Poincaré waves. The solution of Buchwald and Miles (1974) as $\omega \rightarrow 0$ is qualitatively similar.

Wright (1987) reconsidered the TG calculation in the case of the finite basin area, the major complication being that Kelvin waves can propagate completely around the edge of the basin and eventually influence both sides of the strait. Wright finds that geostrophic control is completely expunged from the problem when the basin areas are finite. When one basin is finite and the other semi-infinite, a modified form of geostrophic control does hold in certain frequency ranges.

3. The symmetry principle for hydraulically driven flows

A different approach to the steady flow problem is provided by hydraulic theory. Here the flow is assumed steady at the outset, and the equations of motion are solved with advection terms. Because of the assumption of steadiness we must take the basins to be semi-infinite; otherwise, the water would accumulate in the downstream basin. Furthermore, Wright's (1987) analysis suggests that geostrophic control is applicable primarily in the semi-infinite setting. Hydraulic theory also assumes that the basin/strait geometry varies gradually in the x-direction, so that sharp corners are disallowed.

For simplicity, assume that the strait and basin system of Fig. 1 has a horizontal bottom and that the width $w(x)$ varies on a scale large compared to w itself. If the potential vorticity of the fluid is uniform, the theory of Whitehead et al. (1974) or Gill (1977) specify the possible steady solutions for flow from one basin to the other. As discussed by Gill (1977) some of these solutions possess a symmetry property that can be de-

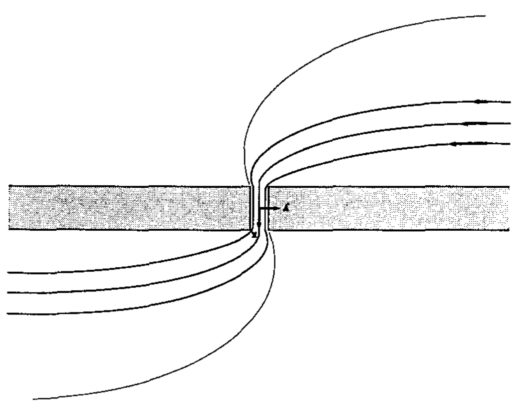

FIG. 2. Low-frequency ($\omega/f \ll 1$) circulation obtained by RC for constant depth.

scribed through consideration of any dependent variable and its variation with x. For example, consider $\zeta(x, o)$, the value of the sea surface elevation along the centerline[1] of the strait. According to hydraulic theory, $\zeta(x, o)$ obeys an equation of the form

$$G[\zeta(x, o), w(x); C_1, C_2, \cdots, C_n] = o \quad (3.1)$$

where the C_n are constants determined by the properties of the flow in the upstream basin (the potential vorticity, for example). Once these constants are specified $\zeta(x, o)$ is determined entirely by the local value of the width. Pratt and Armi (1987) have shown that a relation of the form (3.1) holds for general potential vorticity distributions as well.

Now consider the values ζ_1 and ζ_2 lying along the strait centerline, but well into the interiors of the two basins. If w is the same for each location, the value of ζ_1 determined by (3.1) is identical to ζ_2. Furthermore, if values of $\zeta(x, o)$ can be found for all other widths present in the basin and strait system, a complete solution is obtained. The solution is symmetric, in that for any value of $\zeta(x, o)$ in the left basin there exists an identical $\zeta(x, o)$ in the right basin at the same width value. Many examples of such solutions exist and have been computed; the reader is referred to Gill (1977) for the computational details.

It is clear that any symmetric solution of (3.1) cannot be geostrophically controlled. The basin-to-basin sea level difference $\zeta_1 - \zeta_2$ is zero, while the cross-strait difference is finite. Application of (2.5) would give $Q = 0$, whereas the actual transport is finite.

4. Critical control

The derivation of (3.1) is based, in part, on the fully nonlinear, x-momentum equation

$$uu_x + vu_y - fv = -g\zeta_x. \quad (4.1)$$

The functional G in (3.1) consequently has the nonlinear property that two or more values of $\zeta(x, o)$ may exist for a given $w(x)$. The different roots lie along different solution branches, which can sometimes merge at the narrowest section of the strait. Asymmetric solutions are thus possible, in which case $\zeta(x, o)$ lies along one solution branch in basin 1 and along another in basin 2. These solutions have $\zeta_1 \neq \zeta_2$ and, therefore, could be candidates for geostrophic control.

There are, however, important differences between the asymmetric solutions of hydraulics and those of TG. First, switching of solution branches at the narrowest section is essentially a nonlinear phenomena, whereas the TG and RC solutions are based on a linear theory. Second, it can be shown (Gill 1977 or Pratt and Armi 1987) that the flow at the narrowest section is hydraulically critical (the Kelvin wave speed is zero) when the solution changes branches. Under these conditions, the volume transport Q depends only on the conditions in the upstream basin. Geostrophic control is clearly not applicable since Q cannot depend upon ζ_2.[2] Finally, the transport of a critically controlled flow can be decreased by decreasing the minimum value of the width; the flow can be choked. Choking of geostrophically controlled flow by the same mechanism is clearly impossible since Q does not depend on w [see (2.5)]. These comments suggest that geostrophically controlled flow cannot be achieved in any limit of hydraulic theory, despite the fact that the equations of hydraulics would appear to be more general due to inclusion of nonlinear advection.

5. Conditions for geostrophic control

In order to clearly identify the conditions under which geostrophic control holds, it is helpful to consider the Rossby adjustment problem in a strait (Gill 1976). A barrier placed at $x = 0$ initially separates two resting homogeneous bodies of water (Fig. 3a). The strait has constant width and bottom elevation, and the difference in surface level across the barrier is $\Delta\zeta$. At $t = 0$ the barrier is removed and fluid from the deeper region ($x < o$) is allowed to flow into $x > 0$.

If $\Delta\zeta/h \ll 1$ the linearized shallow water equations (2.6)–(2.8) may be used to obtain a solution valid to $O(\Delta\zeta^2/H^2)$. Gill (1976) performed this calculation and a qualitative sketch of his solution is shown in Fig. 3b, for the case in which the width is equal to four deformation radii (also see his Fig. 3). The removal of the barrier generates Kelvin waves that propagate away from $x = 0$ and set up boundary currents. In Fig. 3b, these waves have propagated about eight deformation radii upstream and downstream of $x = 0$, leaving behind a steady flow in which the fluid approaches $x = 0$ along the left ($y = -w/2$) wall, crosses the strait, and continues along the right ($y = w/2$) wall. The crossing region is set up by Poincaré waves (the Kelvin waves possess no cross-strait velocity). The steady solution is similar in many respects to the low frequency solutions obtained by RC (Fig. 2) and Buchwald and Miles (1974). In particular, the flow is geostrophically controlled, in that the transport is given by (2.5) with $\Delta\zeta$ representing the difference in interior surface level between locations several deformation radii upstream and downstream of $x = 0$. If it were not for weak non-

[1] For simplicity we assume that the channel walls lie at $y = \pm w(x)/2$, although this is not crucial to the overall conclusion.

[2] Strictly speaking it is possible to relate Q to $\zeta_1 - \zeta_2$ using hydraulic theory, even though Q cannot depend upon ζ_1 and ζ_2 individually. However, the predicted flow in the downstream basin, which is hydraulically supercritical, cannot exist in normal physical settings. Instead, the downstream flow is supercritical over a limited distance downstream of the narrowest section and returns to a subcritical state via some type of hydraulic jump (Pratt 1987). Under these conditions the downstream value ζ_2 can be varied while ζ_1 and Q are held fixed.

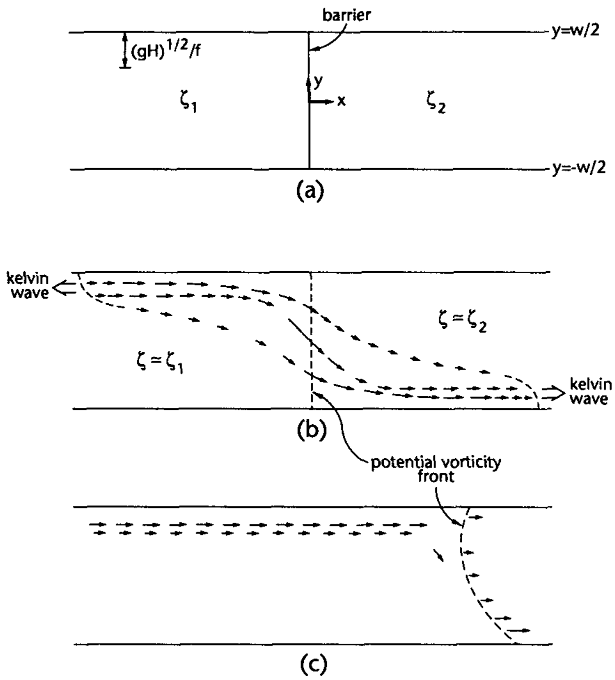

FIG. 3. The Rossby adjustment problem in a channel with (a) the initial state, (b) a qualitative view of the velocity field for $f^{-1} \ll t \ll t_a$ (from Gill 1976), and (c) a qualitative view of the velocity field for $t \gg t_a$ (from Hermann et al. 1989). Here t_a is defined in the text preceding Eq. (5.1).

linear effects that slowly come into play, this solution would be valid for all time.

The breakdown in Gill's (1976) final steady solution occurs due to nonconservation of potential vorticity. The fluid initially lying in $x > 0$ has slightly higher potential vorticity owing to its smaller initial depth and the line $x = 0$ is therefore a potential vorticity front at $t = 0$. In Gill's solution the front remains at $x = 0$ (the dashed line in Fig. 3b) and fluid columns simply experience a potential vorticity increase proportional to $(\Delta\zeta/H)$ as they cross over.

Hermann et al. (1989) has solved the full nonlinear problem and shown that the front is actually advected downstream, leaving behind a *symmetric* flow (Fig. 3c). This new asymptotic state is simply one of the subcritical solutions described by hydraulic theory and is not geostrophically controlled (the drop in interior surface level is zero).

In summary, the geostrophically controlled solution is a temporary state which occurs after Kelvin waves have propagated out of the region in question, but before the vorticity front has been advected out. If we focus on a region lying within a deformation radius of $x = 0$, the Kelvin waves take time f^{-1} to be removed, while the vorticity front is removed after time $(f\Delta\zeta/H)^{-1}\Delta$, where Δ varies from about 1 to 50 as w varies from 0.2 to 10.0 deformation radii (Hermann et al. 1989). Hence, geostrophic control occurs for

$$f^{-1} \ll t \ll (f\Delta\zeta/H)^{-1}\Delta. \qquad (5.1)$$

In the solutions of Buchwald and Miles (1974) and RC, there is a similar mismatch in potential vorticity between basins. As noted by RC, one expects nonlinear effects to come into play over a slow advective time scale, t_a. The Hermann et al. (1989) solution to Gill's adjustment problem suggests that these effects may produce an entirely different steady state than the one shown in Fig. 2, although calculation of such a state is computationally difficult. In any case, we expect geostrophic control to hold for frequencies ω in the range $f \gg \omega \gg t_a^{-1}$.

Hydraulically controlled flows are generally associated with small values of the minimum width-to-basin width ratio and O(1) values of $\Delta\zeta/H$. In the ocean $\Delta\zeta/H$ is typically very small, and the corresponding flows will tend to be completely subcritical rather than hydraulically controlled. Thus, the choice is really one between geostrophically controlled and symmetric subcritical flows. For internal flows, where $\Delta\zeta$ measures the difference in interface or isopycnal level, $\Delta\zeta/H$ can be O(1) and the corresponding flows can be hydraulically controlled.

The above discussion has tacitly ignored friction. Should frictional effects dominate those of advection in the sense that $t_a > 1/\lambda$ (where λ is the linear friction coefficient), then geostrophic control can be expected only within $f^{-1} < t < 1/\lambda$. For $t > O(1/\lambda)$ some frictionally determined steady flow will be established.

As an aside, it is noted that the time-dependent equations of rotating hydraulics will not produce geostrophically controlled flows, even when the time scale is restricted by (5.1). The crossing of the current from one boundary to another is set up by Poincaré waves, and the latter are not allowed in rotating hydraulic theory (which admits only Kelvin wave dynamics). Although nonlinear Kelvin waves possess finite cross-strait velocity, linear Kelvin waves do not. The crossing of the current in the models of RC, Buchwald and Miles (1974), and Gill (1976) is associated with corners, sudden depth changes, or other sources of rapid variations in the x-direction, all of which lead to Poincaré wave dynamics. In the approximations leading to hydraulic theory, rapid along-stream changes are disallowed.

6. Flow between closed basins and Whitehead's (1986) experiment

According to Wright's (1987) calculation, geostrophic control becomes inapplicable when the areas of the upstream and downstream basins are finite. However, Whitehead (1986) has performed an experiment that seems to verify geostrophic control in a laboratory strait connecting two finite basins. A theory is also presented which utilizes elements of hydraulic theory, but which also makes the TG assumption (2.1b) a priori. The resulting transport relation predicts a geostrophically controlled flow when $[2g(\Delta\zeta)]^{1/2}/fW < 1$, where W is the width at the narrowest section.

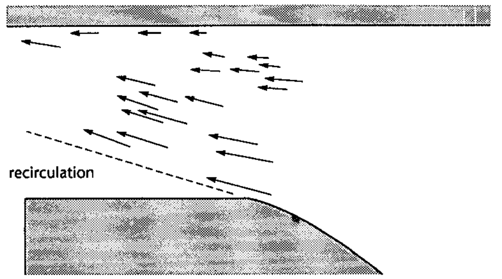

FIG. 4. Sketch of surface velocities and separation streamline (dashed line) in Whitehead's (1986) experiment. (Redrawn from his Fig. 6d).

Measurements of transport for cases of $\Delta\zeta/H \ll 1$ agree within 5% or so with (2.5).

The paradox presented by Wright's (1987) calculation and Whitehead's (1986) experiment can be resolved by referring to a typical plan of Whitehead's laboratory flow reproduced in Fig. 4 from his Fig. 6a. After entering the contraction or strait, the stream separates from the left (facing downstream) wall. The separation streamline is indicated by a dashed line. To the left of the dashed line, a closed and relatively slow moving recirculation exists. The presence of the separation streamline and the relatively stagnant region to its left make the assumption (2.1b) more plausible and may account for the success of geostrophic control theory.

The streamline separation and recirculation observed in Whitehead's experiment are not features of Wright's solution. The conditions producing the separation (the corner geometry, the outlet conditions, side wall friction, etc.) are not identified and no explanation is given for the lack of symmetric subcritical flows (which are observed in other laboratory situations, e.g., Shen 1981). Further study of this situation is clearly needed.

7. Summary

In flow between infinite basins, geostrophic control requires two conditions: first, the change in surface level relative to depth between the basins must be small ($\Delta\zeta/H \ll 1$), a restriction identified by RC and Whitehead (1986). Second, the time span of applicability is limited by

$$f^{-1} < t < t_a$$

where t_a is an advective time scale depending on the particular geometry of the flow. In most cases we expect t_a to be proportional to $f^{-1}H/\Delta\zeta$ and therefore, to be $\gg f^{-1}$. For a periodic flow with frequency ω the corresponding restriction is

$$f \gg \omega \gg t_a^{-1}.$$

For time scales longer than t_a, steady flows are expected to be dominated by advection. In a gradually varying geometry, the solutions of hydraulic theory will arise. Whether the flow will be symmetric (subcritical) or asymmetric (hydraulically controlled) will depend on the initial value of $\Delta\zeta/H$ and the ratio of minimum strait width to basin width. Hydraulically controlled solutions will tend to occur when the former is O(1) and the latter is $\ll 1$. For typical ocean flows, $\Delta\zeta/H$ is tiny and subcritical flow is normally obtained as in the Rossby adjustment problem. For internal flows having an elevation difference $\Delta\zeta$ between isopycnals in neighboring basins, $\Delta\zeta/H$ may be O(1) and hydraulic control is more likely. Frictional effects (tacitly neglected in our discussion) might also determine the steady flow on long time scales.

The situation with regard to finite basins is less clear. Despite theoretical predictions to the contrary, geostrophic control can exist if a peculiar separation phenomenon occurs in the passage. This separation has been observed in the laboratory flow of Whitehead (1986) but not others, and the conditions for its occurrence are not documented.

Acknowledgments. This work was supported by the Office of Naval Research (Contract N00014-89-J-1182) and the National Science Foundation (Grant OCE89-16446). Many of the ideas presented herein were developed in the course of discussions with A. Hermann, D. Wright, A. Clarke, J. Whitehead, and C. Garrett. The author also wishes to thank Lisa Wolfe for preparing the manuscript.

REFERENCES

Buchwald, V. T., and J. W. Miles, 1974: Kelvin-wave diffraction by a gap. *Aust. Math. Soc. J.*, **17**, 29–34.

Garrett, C. J. R., 1983: Variable sea level and strait flows in the Mediterranean: a theoretical study of the response to meteorological forcing. *Oceanol. Acta.*, **6**, 79–87.

——, and B. Toulany, 1982: Sea level variability due to meteorological forcing in the northeast Gulf of St. Lawrence. *J. Geophys. Res.*, **87**, 1968–1978.

——, and F. Majaess, 1984: Nonisostatic response of sea level to atmospheric pressure in the eastern mediterranean. *J. Phys. Oceanogr.*, **14**, 656–665.

Gill, A. E., 1976: Adjustment under gravity in a rotating channel. *J. Phys. Oceanogr.*, **77**, 603–621.

——, 1977: The hydraulics of rotating-channel flow. *J. Fluid Mech.*, **80**, 641–671.

Hermann, A. J., P. B. Rhines and E. R. Johnson, 1989: Nonlinear Rossby adjustment in a channel: beyond Kelvin waves. *J. Fluid Mech.*, **205**, 469–502.

Pratt, L. J., 1987: Rotating shocks in a separated laboratory channel flow. *J. Phys. Oceanogr.*, **17**, 483–491.

——, and L. Armi, 1987: Hydraulic control of flows with nonuniform potential vorticity. *J. Phys. Oceanogr.*, **17**, 2016–2029.

Rocha, C. A., and A. J. Clarke, 1987: Interaction of ocean tides through a narrow single gap and narrow multiple gaps. *J. Phys. Oceanogr.*, **17**, 2203–2218.

Shen, C. Y., 1981: The rotating hydraulics of open-channel flow between two basins. *J. Fluid Mech.*, **112**, 161–188.

Toulany, B., and C. J. R. Garrett, 1984: Geostrophic control of fluctuating flow through straits. *J. Phys. Oceanogr.*, **14**, 649–655.

Whitehead, J. A., 1986: Flow of a homogeneous rotating fluid through straits. *J. Geophys. Astrophys. Fluid Dyn.*, **36**, 187–205.

——, A. Leetmaa and R. A. Knox, 1974: Rotating hydraulics of strait and sill flows. *Geophys. Fluid Dyn.*, **6**, 101–125.

Wright, D. G., 1987: Comments on "Geostrophic control of fluctuating barotropic flow through straits." *J. Phys. Oceanogr.*, **17**, 2375–2377.