Appendix B: The method of characteristics in two dimensions.

(a) Mathematical Theory

Many of the flows dealt with herein involve two independent variables in a two dimensional space. Examples include wave propagation in x and t and steady shallow flow on the (x,y) plane. If we temporarily let x and y represent generic independent variables, then the governing equations take the form:

$$L_1 = A_1 \frac{\partial u}{\partial x} + B_1 \frac{\partial u}{\partial y} + C_1 \frac{\partial v}{\partial x} + D_1 \frac{\partial v}{\partial y} + E_1 = 0 \qquad \text{(B1a)}$$

and

$$L_2 = A_2 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y} + C_2 \frac{\partial v}{\partial x} + D_2 \frac{\partial v}{\partial y} + E_2 = 0\,, \qquad \text{(B1b)}$$

where u and v are generic dependent variables. For the systems we consider, the coefficients A_1, A_2, etc. may depend on x, y, u and v, but not on the derivatives of u and v. The governing equations are then *quasilinear* and may be amenable to solution using the method of characteristics, provided that further conditions hold. The development of this approach is laid out in Courant and Friedrichs (1948) and the following summary is based on their notation and expose′.

In general, we wish to take advantage of the physical property of certain systems that all information propagates in a 'forward' direction, usually meaning positive x or y, and at finite speed. Such flows are called *locally supercritical*.[1] We should then be able to construct solutions by forward integration along the paths of information travel beginning from the boundaries at which the information is generated. To make these ideas more precise, consider a path $x=x(\sigma)$ and $y=y(\sigma)$, parameterized by the variable σ. The vector $(dx/d\sigma, dy/d\sigma)$ is tangent to the path and the derivative of a function f in the same direction and with respect to σ is $df/d\sigma = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \sigma}$. The first aim of the analysis is to manipulate (B1) to form a single equation in which the x- and y- derivatives of u and v combine to form derivatives in a particular direction. This *characteristic* direction depends on u, v, x, and y and defines a *characteristic curve* along which the derivative is taken. To this end, take $L = \lambda_1 L_1 + \lambda_2 L_2$, leading to

$$L = (\lambda_1 A_1 + \lambda_2 A_2)\frac{\partial u}{\partial x} + (\lambda_1 B_1 + \lambda_2 B_2)\frac{\partial u}{\partial y} + (\lambda_1 C_1 + \lambda_2 C_2)\frac{\partial v}{\partial x} + (\lambda_1 D_1 + \lambda_2 D_2)\frac{\partial v}{\partial y} + (\lambda_1 E_1 + \lambda_2 E_2) = 0$$

.

[1] This term should be distinguished from *hydraulically supercritical*. The latter describes a flow in which normal modes, which are felt across the breadth of the flow and which satisfies the boundary conditions at the edges, propagate in a single direction. Hydraulically supercritical flows need not be locally supercritical across their entire width.

(B2)

In order that the derivatives of u and v along the hypothetical path be the same, we need

$$\frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2} = \frac{\lambda_1 C_1 + \lambda_2 C_2}{\lambda_1 D_1 + \lambda_2 D_2} = \frac{\partial x / \partial \sigma}{\partial y / \partial \sigma}, \qquad \text{(B3)}$$

which allows (B2) to be written as

$$L = (\lambda_1 A_1 + \lambda_2 A_2)\left[\frac{\partial u}{\partial x} + \frac{\partial y/\partial\sigma}{\partial x/\partial\sigma}\frac{\partial u}{\partial y}\right] + (\lambda_1 C_1 + \lambda_2 C_2)\left[\frac{\partial v}{\partial x} + \frac{\partial y/\partial\sigma}{\partial x/\partial\sigma}\frac{\partial v}{\partial y}\right] + (\lambda_1 E_1 + \lambda_2 E_2) = 0\ ,$$

or, using $\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial f}{\partial y}\frac{\partial x}{\partial \sigma}$:

$$L\frac{\partial x}{\partial \sigma} = (\lambda_1 A_1 + \lambda_2 A_2)\frac{\partial u}{\partial \sigma} + (\lambda_1 C_1 + \lambda_2 C_2)\frac{\partial v}{\partial \sigma} + (\lambda_1 E_1 + \lambda_2 E_2)\frac{\partial x}{\partial \sigma} = 0\,. \qquad \text{(B4a)}$$

By a similar approach

$$L\frac{\partial y}{\partial \sigma} = (\lambda_1 B_1 + \lambda_2 B_2)\frac{\partial u}{\partial \sigma} + (\lambda_1 D_1 + \lambda_2 D_2)\frac{\partial v}{\partial \sigma} + (\lambda_1 E_1 + \lambda_2 E_2)\frac{\partial y}{\partial \sigma} = 0. \qquad \text{(B4b)}$$

The factors λ_1 and λ_2 are determined by rearranging (B3) as

$$\lambda_1 (A_1 \frac{\partial y}{\partial \sigma} - B_1 \frac{\partial x}{\partial \sigma}) + \lambda_2 (A_2 \frac{\partial y}{\partial \sigma} - B_2 \frac{\partial x}{\partial \sigma}) = 0 \qquad \text{(B5a)}$$

and

$$\lambda_1 (C_1 \frac{\partial y}{\partial \sigma} - D_1 \frac{\partial x}{\partial \sigma}) + \lambda_2 (C_2 \frac{\partial y}{\partial \sigma} - D_2 \frac{\partial x}{\partial \sigma}) = 0\,. \qquad \text{(B5b)}$$

Setting the determinant of the coefficients of λ_1 and λ_2 to zero leads to

$$a\left(\frac{dy}{d\sigma}\right)^2 - 2b\frac{dy}{d\sigma}\frac{dx}{d\sigma} + c\left(\frac{dx}{d\sigma}\right)^2 = 0\,, \qquad \text{(B6)}$$

where

$$a = [AC],\ \ 2b = [AD] + [BC],\ \ c = [BD], \qquad \text{(B7)}$$

and $[MN]=M_1N_2-M_2N_1$.

With $(dy/d\sigma)/(dx/d\sigma)=dy/dx$, the *characteristic direction* (dx,dy) is given by

$$a\left(\frac{dy}{dx}\right)^2-2b\frac{dy}{dx}+c=0\,.$$

This equation has two distinct real solutions $(dy/dx)_-$ and $(dy/dx)_+$ if and only if

$$b^2>ac. \tag{B8}$$

If (B8) is satisfied, the governing equations are called *hyperbolic*. If the equations are hyperbolic within a finite region of the (x,y) plane, then two distinct characteristic curves C_- and C_+ can be found within this region. The curves are computed from

$$\left(\frac{dy}{dx}\right)_{\pm}=\frac{b\pm\sqrt{b^2-ac}}{a}\,. \tag{B9}$$

We will now use σ_- and σ_+ (formerly σ) to parameterize the two characteristic curves. Thus, a curve determined by the '+' sign in (B9) has σ_-=constant, and vice versa. The original intent was to obtain a form of the governing equations in which derivatives are taken in a characteristic direction, i.e. along one of the characteristic curves. Either of (B4a) or (B4b) provides a basis for the desired result, but λ_1 and λ_2 must first be eliminated. If one attempts to do so using (B4a) and (B5a), say, then it follows that

$$\begin{vmatrix} A_1\partial y/\partial\sigma_\pm-B_1\partial x/\partial\sigma_\pm & A_2\partial y/\partial\sigma_\pm-B_2\partial x/\partial\sigma_\pm \\ A_1\partial u/\partial\sigma_\pm+C_1\partial v/\partial\sigma_\pm+E_1\partial x/\partial\sigma_\pm & A_2\partial u/\partial\sigma_\pm+C_2\partial v/\partial\sigma_\pm+E_2\partial x/\partial\sigma_\pm \end{vmatrix}=0\,.$$

or

$$T\frac{\partial u}{\partial\sigma_\pm}+\left(a\left(\frac{dy}{dx}\right)_\pm-S\right)\frac{\partial v}{\partial\sigma_\pm}+\left(K\left(\frac{dy}{dx}\right)_\pm-H\right)\frac{\partial x}{\partial\sigma_\pm}=0\,, \tag{B10}$$

where

$T=[AB]$, $S=[BC]$, $K=[AE]$, and $H=[BE]$.

A useful alternative to (B10) can be obtained by eliminating λ_1 and λ_2 between (B4a,b):

$$\begin{vmatrix} A_1\partial u/\partial\sigma_\pm + C_1\partial v/\partial\sigma_\pm + E_1\partial x/\partial\sigma_\pm & A_2\partial u/\partial\sigma_\pm + C_2\partial v/\partial\sigma_\pm + E_2\partial x/\partial\sigma_\pm \\ B_1\partial u/\partial\sigma_\pm + D_1\partial v/\partial\sigma_\pm + E_1\partial y/\partial\sigma_\pm & B_2\partial u/\partial\sigma_\pm + D_2\partial v/\partial\sigma_\pm + E_2\partial y/\partial\sigma_\pm \end{vmatrix} = 0\,. \quad \text{(B11)}$$

(b) Example 1: Steady, irrotational, two-dimensional, shallow flow over a horizontal bottom.

We use (x,y), (u,v) and d to denote the nondimensional position, velocity and depth variables, as defined in Section 2.1. Dimensional versions of the following relations may be obtained by replacing d by gd, where g is the gravitational acceleration. The flow to be considered is governed by the continuity equation

$$\frac{\partial(ud)}{\partial x} + \frac{\partial(vd)}{\partial y}\,, \qquad \text{(B12)}$$

and by the statmement of conservation of energy

$$\frac{u^2+v^2}{2} + d = B\,. \qquad \text{(B13)}$$

Although the Bernoulli function B is normally a function of the streamfunction, the assumption of irrotational flow (zero vorticity) renders it a constant.

If the gradient of (B13) is used to eliminate the gradient of d from (B12), one obtains

$$(d-u^2)\frac{\partial u}{\partial x} - uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (d-v^2)\frac{\partial v}{\partial y} = 0\,. \qquad \text{(B14)}$$

Together with the condition of zero vorticity:

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0\,, \qquad \text{(B15)}$$

(B14) forms a system of two quasilinear equations of the form (B1) with A_1=d-u^2, B_1=C_1=-uv, D_1=d-v^2, B_2=-C_2=1, and A_2=D_2=E_2=E_1=0. The two dependent variables are u and v, with d regarded as a function of u and v through (B13). With these coefficients we have a=(u^2-d), b=uv, c=(v^2-d), and thus the condition for hyperbolicity (B8) is

$$u^2 + v^2 > d\,. \qquad \text{(B16).}$$

Equation (B6) governing the characteristic curves is now

$$(d-u^2)\left(\frac{dy}{dx}\right)_\pm^2+2uv\left(\frac{dy}{dx}\right)_\pm+(d-v^2)=0\,, \qquad \text{(B17)}$$

whereas (B11) yields

$$(d-u^2)\left(\frac{du}{dv}\right)_\pm^2-2uv\left(\frac{du}{dv}\right)_\pm+(d-v^2)=0\,. \qquad \text{(B18)}$$

Keep in mind that *du* and *dv* represent changes in *u* and *v* measured along the curve.

A convenient expression for the orientation of the characteristic curves can be determined by rewriting (B17), in the form

$$d=\frac{(udy-vdx)^2}{dx_\pm{}^2+dy_\pm{}^2}=\frac{[\mathbf{k}\cdot(u,v)\times(dx_\pm,dy_\pm)]^2}{dx_\pm^2+dy_\pm^2}=\frac{|(u,v)|^2\,|dx_\pm,dy_\pm|^2}{dx_\pm^2+dy_\pm^2}\sin^2 A\,,$$

or

$$(d)^{1/2}=|(u,v)|\sin(\pm A)\,, \qquad \text{(B19)}$$

In other words, the characteristic curves at any point form an angle $A=\pm\sin^{-1}[d/(u^2+v^2)^{1/2}]$ with respect to the local flow direction (streamline). *A* is analogous to the Mach angle of gas dynamics, named after Ernst Mach. In shallow water theory, *A* is sometimes referred to as the Froude angle.

It is often helpful to consider the images of the (*x*,*y*)-plane characteristics in the (*u*,*v*)-plane, often called the *hodograph*. To this end, note that (B17) and (B18) together imply

$$\left(\frac{dy}{dx}\right)_\pm=-\left(\frac{du}{dv}\right)_\mp \qquad \text{(B20)}$$

[The coordination between the '+' and '-' subscripts must be established independently and can be done so using equations (B4) and (B5).] Now consider a pair of '+' and '-' characteristic curves C_+ and C_- that intersect at some point P in the (*x*,*y*)-plane (Figure 4.4.1a). If α and β are used as parameters along C_+ and C_-, then

$$\frac{\partial y}{\partial\alpha}=\left(\frac{dy}{dx}\right)_+\frac{\partial x}{\partial\alpha}\text{ along }C_+\text{, and}\qquad \frac{\partial y}{\partial\beta}=\left(\frac{dy}{dx}\right)_+\frac{\partial x}{\partial\beta}\text{ along }C_-. \qquad \text{(B21)}$$

The velocity at P determines a point in the (u,v)-plane through which the images Γ_+ and Γ_- of C_+ and C_- pass. According to (B20)

$$\frac{\partial u}{\partial \alpha} = -\left(\frac{dy}{dx}\right)_- \frac{\partial v}{\partial \alpha} \text{ along } C_+, \text{ and} \qquad \frac{\partial u}{\partial \beta} = -\left(\frac{dy}{dx}\right)_- \frac{\partial v}{\partial \beta} \text{ along } C_-. \quad \text{(B22)}$$

It follows that $\frac{\partial u}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial v}{\partial \alpha}\frac{\partial y}{\partial \beta} = 0$ and $\frac{\partial u}{\partial \beta}\frac{\partial x}{\partial \alpha} + \frac{\partial v}{\partial \beta}\frac{\partial y}{\partial \alpha} = 0$, and thus C_+ is perpendicular to the image Γ_- of C_-, and vice versa, if the two are plotted in the same space.

The geometry of the characteristics and their images can be summarized as follows. The two characteristic curves C_+ and C_- passing through P form Froude angle $A = \pm\sin^{-1}[d/(u^2+v^2)^{1/2}]$ with respect to the streamline that passes through P (Figure 4.4.1a). If plotted in the same space the hodograph image Γ_+ of C_+ forms a right angle with C_-, and vice versa, at P (Figure 4.4.1b). The relationship between A and the angle A' in the (u,v) plane between characteristics and streamlines is thus

$$A' = 90^\circ - A. \qquad \text{(B23)}$$

It follows from (B19) that

$$d^{1/2} = |(u,v)|\cos(A'). \qquad \text{(B24)}$$

For computational purposes, it is convenient to introduce the angle θ between the streamline and the x-axis (Figure 4.4.1a):

$$u = q\cos\theta \text{ and } v = q\sin\theta. \qquad \text{(B25)}$$

It follows that

$$\left(\frac{dy}{dx}\right)_\pm = \tan(\theta \pm A), \qquad \text{(B26)}$$

where we have introduced the convention that C_+ tilts to the left, and C_- to the right, as seen by an observer facing downstream. In these terms, equations (B21) and (B22) become

$$\cos(\theta + A)\frac{\partial y}{\partial \alpha} = \sin(\theta + A)\frac{\partial x}{\partial \alpha} \text{ along } C_+,$$

$$\cos(\theta - A)\frac{\partial y}{\partial \beta} = \sin(\theta - A)\frac{\partial x}{\partial \beta} \text{ along } C_-,$$

$$\sin(\theta - A)\frac{\partial v}{\partial \alpha} = -\cos(\theta - A)\frac{\partial u}{\partial \alpha} \text{ along } \Gamma_+, \qquad \text{(B25)}$$

$$\sin(\theta + A)\frac{\partial v}{\partial \beta} = -\cos(\theta + A)\frac{\partial u}{\partial \beta} \text{ along } \Gamma_-.$$

This set could form the basis for a numerical calculation in which characteristic curves emerge from a boundary along which *u* and *v* are known. The paths of the curves penetrating into the domain of interest are calculated by solving (B25) simultaneously.

(c) Example 2: One-dimensional, time-dependent shallow flow over a horizontal bottom

In this example we follow the Chapter 1 notation convention that un-starred variables are dimensional. The dimensional governing equations (2.1.1) and (2.2.2), with *h*=constant, can be expressed in the form (B1) with (t,x) in place of (x,y), *d* in place of *v*, and $A_1=C_2=1$, $B_1=D_2=u$, $B_2=d$, $D_1=g$, and $C_1=A_2=E_1=E_2=0$. Equation (B9) then gives

$$\left(\frac{dy}{dx}\right)_\pm = u - (gd)^{1/2},$$

while the characteristic equations, obtained using (B10), are

$$\frac{\partial u}{\partial \sigma_\pm} \pm \left(\frac{g}{d}\right)^{1/2} \frac{\partial d}{\partial \sigma_\pm} = 0\,.$$

The latter can also be written in the form $\partial R_\pm / \partial \sigma_\pm = 0$, where $R_\pm = u \pm (gd)^{1/2}$ is the Riemann invariant.