Appendix C: The method of characteristics for steady, 2-d, shallow flow with vorticity.

The shallow-water equations for steady flow over a horizontal bottom can be written in the nondimensional forms

$$L_1 = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial d}{\partial x} - v = 0\,, \qquad \text{(C1a)}$$

$$L_2 = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial d}{\partial y} + u = 0\,, \qquad \text{(C1b)}$$

$$L_3 = d\frac{\partial u}{\partial x} + d\frac{\partial v}{\partial y} + u\frac{\partial d}{\partial x} + v\frac{\partial d}{\partial y} = 0\,. \qquad \text{(C1c)}$$

The additional dependent variable *d* is the depth of the flow. In contrast to the cases discussed in Appendix B, the system contains three dependent variables and this number cannot be reduced without further assumptions. A characteristic form can nevertheless be sought using the same reasoning; one would like to combine the constituent equations linearly such that differentiation of each variable takes place in a single direction. Taking the combination $L = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3$ leads to

$$\begin{aligned} L &= (\lambda_1 u + \lambda_3 d)\frac{\partial u}{\partial x} + \lambda_1 v\frac{\partial u}{\partial y} \\ &+\lambda_2 u\frac{\partial v}{\partial x} + (\lambda_2 v + \lambda_3 d)\frac{\partial v}{\partial y} \\ &+(\lambda_1 + \lambda_3 u)\frac{\partial d}{\partial x} + (\lambda_2 + \lambda_3 v)\frac{\partial d}{\partial y} \\ &-\lambda_1 v + \lambda_2 u = 0. \end{aligned} \qquad \text{(C2)}$$

If the direction of differentiation of *u*, *v*, and *d* is along the curve $(x(\sigma), y(\sigma))$ then it is necessary that

$$\frac{(\lambda_1 u + \lambda_3 d)}{\lambda_1 v} = \frac{\lambda_2 u}{\lambda_2 v + \lambda_3 d} = \frac{\lambda_1 + \lambda_3 u}{\lambda_2 + \lambda_3 v} = \frac{\partial x / \partial\sigma}{\partial y / \partial\sigma}\,. \qquad \text{(C3)}$$

The characteristic directions are then determined by the solvability condition for (C3). Writing the three equations in the form

$$\begin{pmatrix} u\dfrac{\partial y}{\partial\sigma}-v\dfrac{\partial x}{\partial\sigma} & 0 & d\dfrac{\partial y}{\partial\sigma} \\ 0 & u\dfrac{\partial y}{\partial\sigma}-v\dfrac{\partial x}{\partial\sigma} & -d\dfrac{\partial x}{\partial\sigma} \\ \dfrac{\partial y}{\partial\sigma} & -\dfrac{\partial x}{\partial\sigma} & u\dfrac{\partial y}{\partial\sigma}-v\dfrac{\partial x}{\partial\sigma} \end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}=0$$

and setting the determinant of the coefficient matrix to zero leads to

$$\left(u\frac{\partial y}{\partial\sigma}-v\frac{\partial x}{\partial\sigma}\right)\left[\left(u\frac{\partial y}{\partial\sigma}-v\frac{\partial x}{\partial\sigma}\right)^2-d\left(\left(\frac{\partial x}{\partial\sigma}\right)^2+\left(\frac{\partial y}{\partial\sigma}\right)^2\right)\right]=0\,. \qquad \text{(C4)}$$

One of the characteristic directions is the flow direction itself:

$$\frac{\partial y/\partial\sigma}{\partial x/\partial\sigma}=\frac{v}{u}. \qquad \text{(C5)}$$

The corresponding characteristic curves are just the streamlines of the flow and the characteristic equation expresses the conservation of the Bernoulli function along this path. [This constraint is absent in the *irrotational* case discussed in Example 1 of Appendix B because the Bernoulli function is uniform throughout the domain.] We will denote these curves using the subscript ψ.

The remaining two characteristic directions are obtained by setting the bracketed term in (C4) to zero. Rearrangement of this expression leads to (B19), and thus the second and third characteristic directions are the same as for the case of irrotational flow. The corresponding characteristic curves are again denoted C_+ and C_- with corresponding parameters σ_+ and σ_-. These curves cross streamlines at the Froude angle $\pm A$, where $d=|(u,v)|^2\sin^2 A$, as shown in Figure 4.4.1a. The characteristic equations are obtained by multiplying (C2) by $\partial x/\partial\sigma_\pm$, leading to

$$\frac{\partial x}{\partial\sigma_\pm}L_1=(\lambda_1 u+\lambda_3 d)\frac{\partial u}{\partial\sigma_\pm}+\lambda_2 u\frac{\partial v}{\partial\sigma_\pm}+(\lambda_1+\lambda_3 u)\frac{\partial d}{\partial\sigma_\pm}+\left(\lambda_2 u-\lambda_1 v\right)\frac{\partial x}{\partial\sigma_\pm} \qquad \text{(C6)}$$

$$=\frac{\lambda_3}{u\dfrac{\partial y}{\partial\sigma_\pm}-v\dfrac{\partial x}{\partial\sigma_\pm}}\left\{\begin{array}{l} d\left(u\dfrac{\partial v}{\partial\sigma_\pm}-v\dfrac{\partial u}{\partial\sigma_\pm}\right)\dfrac{\partial x}{\partial\sigma_\pm}+ \\ \qquad\left[u\left(u\dfrac{\partial y}{\partial\sigma_\pm}-v\dfrac{\partial x}{\partial\sigma_\pm}\right)-d\dfrac{\partial y}{\partial\sigma_\pm}\right]\dfrac{\partial d}{\partial\sigma_\pm}+d\left(u\dfrac{\partial x}{\partial\sigma_\pm}+v\dfrac{\partial y}{\partial\sigma_\pm}\right)\dfrac{\partial x}{\partial\sigma_\pm}\end{array}\right\}=0.$$

The relationships $\lambda_1 = \dfrac{-d\dfrac{\partial y}{\partial \sigma_\pm}}{u\dfrac{\partial y}{\partial \sigma_\pm} - v\dfrac{\partial x}{\partial \sigma_\pm}}\lambda_3$ and $\lambda_2 = \dfrac{d\dfrac{\partial x}{\partial \sigma}}{u\dfrac{\partial y}{\partial \sigma_\pm} - v\dfrac{\partial x}{\partial \sigma_\pm}}\lambda_3$, both obtained from (C3) have been used to obtain the second step. A convenient form of the characteristic equation is obtained by writing (C6) in terms of the variables $|u|$ and θ, where $(u,v) = |u|(\cos\theta, \sin\theta)$. As shown in Figure 4.4.1a, the characteristic curves C_+ and C_- are inclined at the angle $\theta \pm A$ with respect to the x-axis and

$$u\frac{\partial x}{\partial \sigma_\pm} + v\frac{\partial y}{\partial \sigma_\pm} = |\mathbf{u}|\left|\frac{d\mathbf{x}}{d\sigma_\pm}\right|\cos A, \tag{C7a}$$

$$u\frac{\partial y}{\partial \sigma_\pm} - v\frac{\partial x}{\partial \sigma_\pm} = \pm|\mathbf{u}|\left|\frac{d\mathbf{x}}{d\sigma_\pm}\right|\sin A \tag{C7b}$$

and

$$u\frac{\partial v}{\partial \sigma_\pm} - v\frac{\partial u}{\partial \sigma_\pm} = |\mathbf{u}|^2\frac{d\theta}{d\sigma_\pm}, \tag{C7c}$$

where

$$\left|\frac{d\mathbf{x}}{d\sigma_\pm}\right| = \left[\left(\frac{\partial x}{\partial \sigma_\pm}\right)^2 + \left(\frac{\partial y}{\partial \sigma_\pm}\right)^2\right]^{1/2}$$

Use of (C7) in the second equality of (C6) and division of the result by $d|\mathbf{u}|(\partial x / \partial \sigma_\pm)$ leads to

$$|\mathbf{u}|\frac{\partial \theta}{\partial \sigma_\pm} + \frac{1}{|\mathbf{u}|d}C_L\frac{\partial d}{\partial \sigma_\pm} + \left|\frac{d\mathbf{x}}{d\sigma_\pm}\right|\cos A = 0 \quad (\text{on } C_\pm). \tag{C8}$$

where

$$C_L = \frac{\left[\pm u|\mathbf{u}|\left|\dfrac{d\mathbf{x}}{d\sigma_\pm}\right|\sin A - d\dfrac{\partial y}{\partial \sigma_\pm}\right]}{\dfrac{\partial x}{\partial \sigma_\pm}} = \frac{\left[\pm v|\mathbf{u}|\left|\dfrac{d\mathbf{x}}{d\sigma_\pm}\right|\sin A + d\dfrac{\partial x}{\partial \sigma_\pm}\right]}{\dfrac{\partial y}{\partial \sigma_\pm}}, \tag{C9}$$

and where the second equality follows from (A19). The term C_L can be further simplified by writing it as the linear combination $C_L = \alpha C_L + \beta C_R$ where C_R represents the final expression in (C9), $\alpha+\beta=1$, and α is chosen so that the terms proportional to $|\mathbf{u}|$ cancel. Thus

$$|\mathbf{u}|\frac{\partial\theta}{\partial\sigma_\pm} \pm \frac{\cos A}{|\mathbf{u}|\sin A}\frac{\partial d}{\partial\sigma_\pm} + \left|\frac{d\mathbf{x}}{d\sigma_\pm}\right|\cos A = 0 \quad (\text{on } C_\pm) \tag{C10}$$

If $\sigma_\pm$ is chosen to be arclength measured along the characteristic curve, then $|d\mathbf{x}/d\sigma_\pm = 1|$.

A more traditional form of (C10), one having roots in the field of aerodynamics, uses the intrinsic long wave speed $d^{1/2}$ as a variable. The middle term in (C10) would then be written

$$\pm\frac{\cos A}{|\mathbf{u}|\sin A}\frac{\partial d}{\partial\sigma_\pm} = \pm 2\frac{d^{1/2}\cos A}{|\mathbf{u}|\sin A}\frac{\partial (d)^{1/2}}{\partial\sigma_\pm} = 2\cos A\frac{\partial (d)^{1/2}}{\partial\sigma_\pm}, \tag{C11}$$

where (B19) has been used in the last step.

In summary, the three sets of characteristic curves include the streamlines (see C5) and the curves that cross the streamlines at the Froude angle $\pm A$ (see B19). The characteristic equation that applies along streamlines is simply the Bernoulli equation:

$$\frac{|\mathbf{u}|^2}{2} + gd = B(\psi), \tag{C12}$$

while the equations that apply along the Froude lines are given by (C8) or (C10). The dimensional versions these three can be obtained by replacing d by gd and $\left|\frac{d\mathbf{x}}{d\sigma_\pm}\right|$ by $f\left|\frac{d\mathbf{x}}{d\sigma_\pm}\right|$, where f is the Coriolis parameter representing the background rotation.

An example of the use of these relations to compute a supercritical coastal current emerging from a river mouth is given by Garvine (1987).[1]

[1] A misprint in this reference lists θ in place of A in the middle term of (C10). The calculations presented are based on the correct formulation, however.