## 1.11 Nonlinear Dispersion.

The hydrostatic approximation is central to everything discussed to this point. Nonhydrostatic effects associated with vertical accelerations of the fluid remain small as long as the ratio of the depth scale to horizontal length scale is small. However, special circumstances may exist that allow nonhydrostatic effects to become important even when this scale separation exists. For example, when Long's experiment is performed for $h_m/d_o$<<1 and $F_o \cong 1$, wave-like free surface effects arise in certain parts of the flow filed. In fact, certain values of $h_m/d_o$ and $F_o$ produce a situation in which the flow refuses to settle into a steady state (Baines, 1995, Sec. 2.4). Another example is a hydraulic jump with an upstream Froude number less than about 1.7. Instead of an abrupt transition between supercritical and subcritical flow extending over a few depth scales, the jump is undular and extends over a much longer distance.

In both of these examples, changes in $v$ and $d$ along the channel are relatively small and corresponding advective terms like $v\partial v/\partial y$ are weak. Small terms like $w\partial v/\partial z$, which have been neglected as a consequence of the nonhydrostatic approximation, may now be as large as the retained terms. The conservation laws for momentum and mass now consist of delicate balances between weak hydrostatic and nonhydrostatic terms. Some insight into the form of the governing equation can be gained through consideration of the special case of a wave propagating into an undisturbed fluid with uniform depth $D$ and positive velocity $V$. Let us assume that the wave attempts to propagates against the current so that, within the context of shallow water theory, its evolution is governed by (1.3.1):

$$\left\{\frac{\partial}{\partial t}+[v-(gd)^{1/2}]\frac{\partial}{\partial y}\right\}[v-2(gd)^{1/2}]=0 \qquad (1.11.1)$$

The value of the Riemann invariant $R_+=v+2(gd)^{1/2}$ is equal to its value $V+2(gD)^{1/2}$ in the undisturbed fluid and thus $v$ in the above equation can be replaced by $V-2(gd)^{1/2}+2(gD)^{1/2}$. If the depth in the wave is only slightly different than $D$, we can write $d=D+\eta$, where $\eta/D$<<1. Then $(gd)^{1/2}=(gD)^{1/2}[1+(\eta/2D)+\cdots]$ and substitution into (1.11.1) yields

$$\left\{\frac{\partial}{\partial t}+c_o\left[1-\frac{3\eta}{2D}\right]\frac{\partial}{\partial y}\right\}\eta+O\left(\frac{\eta}{D}\right)^3=0\,, \qquad (1.11.2)$$

where $c_o=V-(gD)^{1/2}$

The correction introduced into this equation by nonhydrostatic effects can be anticipated through consideration of the dispersion relation $\omega=[gl\tanh(lD)]^{1/2}$ for a surface gravity wave propagating in a resting fluid of uniform depth $D$. The wave has the form $\eta = ae^{i(ly-\omega t)}$, where $\omega$ denotes the frequency and $2\pi/l$ the (arbitrarily short) wavelength. If the latter is long compared to $D$, this relation may be expanded:

$$\omega = Vl - [gl\tanh(lD)]^{1/2} = Vl - (gD)^{1/2} l[1 - \frac{(ld)^2}{6} + O(ld)^4]$$

The linear equation that would produce the two leading terms in this expansion is

$$\left[\frac{\partial}{\partial t} + c_o \frac{\partial}{\partial y}\right]\eta + \frac{(gD)^{1/2} D^2}{6}\frac{\partial^3 \eta}{\partial y^3} = 0$$

and thus the nonhydrostatic correction should be $-\frac{1}{6}(gD)^{1/2} D^2 \partial^3 \eta / \partial y^3$.

If (1.11.2) is modified to include this factor, the result is the celebrated *Korteweg-de Vries* (KdV) *equation*

$$\left\{\frac{\partial}{\partial t} + \left[c_o - \frac{3g^{1/2}\eta}{2D^{1/2}}\right]\frac{\partial}{\partial y}\right\}\eta - \frac{(gD)^{1/2} D^2}{6}\frac{\partial^3 \eta}{\partial y^3} = 0\,, \qquad (1.11.3)$$

as can be verified by a more careful analysis (Whitham, 1974).

According to (1.11.3), the wave propagates at the base speed $c_o$ and evolves slowly in response to weak nonlinearity and dispersion. The competition between the two processes can be isolated by expressing the equation in frame of reference moving at the base speed. With $y' = y - c_o t$, we have

$$\left\{\frac{\partial}{\partial t} - \frac{3g^{1/2}\eta}{2D^{1/2}}\frac{\partial}{\partial y'}\right\}\eta - \frac{(gD)^{1/2} D^2}{6}\frac{\partial^3 \eta}{\partial y'^3} = 0\,. \qquad (1.11.4)$$

It is possible to find steadily-propagating solutions, one of which is the soliton:

$$\eta = \eta_o \mathrm{sech}^2\left[\frac{3\eta_o}{4D^2}(y + \hat{c}t)\right]$$

where $\hat{c} = c_o\left(1 + \frac{\eta_o}{2D}\right)$. In this case a balance between steepening and dispersion has achieved an isolated disturbance of permanent form that propagates with an amplitude-dependent speed. A class of periodic disturbances ('Cnoidal waves') is also admitted, as explored in Exercise 1.

The KdV equation and its extensions have been successfully used in the analysis of undular bores (Peregrine, 1966 and Fornberg and Whitham 1978). For hydraulic applications, a topographically forced version of (1.11.4) may be used. To remain consistency with the assumption of unidirectional propagation, any forcing that is added must move at the base speed $c_o$ of the disturbance. Stationary forcing therefore requires that $c_o$ is zero: that is, the flow is critical to leading order. The obstacle height must also be small in order to preserve consistency with the assumption of weak nonlinearity. The evolution equation is then obtained through introduction of the term $\frac{1}{2}(gD)^{1/2}dh/dy$ on the right hand side of (1.11.4). Long's towing problem with $h_m/d_o \ll 1$ and $F_o \cong 1$ (Cole, 1985 and Grimshaw and Smyth, 1986), and other hydraulic applications. The reader is referred to Baines 1995 for a more thorough summary.

One interesting and simple application is to the problem of steady, shallow flow over consecutive obstacles of identical height (Figure 1.11.1). According to shallow water theory, there is no solution that is hydraulically controlled and everwhere stable. If the approach subcritical (solution *ab* in the figure), a subcritical-to-supercritical transition occurs over the first sill. The approach to the second obstacle is now supercritical and an (unstable) transition back to a subcritical state is required. This transition is shown as a dashed section of the *ab* curve. It is also possible for the flow approaching the first obstacle to be supercritical (solution *cd*) but then an unstable transition is forced over the first sill. Nor is it possible to avoid the unstable transitions by introducing a hydraulic jump between the obstacles: the resulting energy loss would prevent the flow from surmounting the next obstacle. It would seem, then, that shallow water theory fails to provide a satisfactory steady solution.

Laboratory experiments (Figure 1.11.2) have shown, in fact, that the spilling flow occurs over the second obstacle and that the flow between the two obstacles is wavelike. The heights of the obstacles do not need to be identical for this behavior to occur, and the phenomena appears to be more than just a curiosity. A solution with the observed properties can be found to the forced KdV equation for nearly critical flow. The reader is referred to Exercise 1 for more details.

**Exercises**

1) As described in the text, the equation governing steady, weakly nonlinear, weakly dispersive flow over a small obstacle is obtained by setting $c_o$=0 and adding the term $\frac{1}{2}(gD)^{1/2}dh/dy$ to the right-hand side of the steady form of (1.11.3). The result is

$$\frac{3g^{1/2}\eta}{2D^{1/2}}\frac{d\eta}{dy}+\frac{(gD)^{1/2}D^2}{6}\frac{d^3\eta}{dy^3}+\frac{(gD)^{1/2}}{2}\frac{dh}{dy}=0\,. \qquad (1.11.5)$$

(a) Integrate (1.11.5) once and show that the result can be written as the following dynamical system:

$$\frac{d\zeta}{dy} = C - \frac{9}{2}\hat{\eta}^2 - 2\hat{h} \tag{1.11.6}$$

and

$$\frac{d\hat{\eta}}{dy} = \hat{\zeta}\,, \tag{1.11.7}$$

where $C$ is a constant.

(b) For the case of a flat bottom, note that for each $C$>0 there are two uniform ($d/dy$=0) flows corresponding to $\hat{\zeta} = 0$ and $\hat{\eta}_{\pm} = \pm\sqrt{2C/9}$. Draw a picture of the phase plane $(\hat{\zeta}, \hat{\eta})$ and locate the points corresponding to the two uniform flows. Show the solutions trajectories near $(0, \hat{\eta}_+)$ are closed, corresponding to a set of stationary periodic waveforms. These are the 'cnoidal' waves referred to in the text. Also show that the solution $(0, \hat{\eta}_-)$ is unstable, corresponding to trajectories that diverge away.

(c) Also show that one of the trajectories that diverges from $(0, \hat{\eta}_-)$ forms a closed obit that circumnavigates $(0, \hat{\eta}_+)$. This solution corresponds to a stationary solitary wave.

The addition of the topography term in (1.11.6) allows the actual solution to cross the trajectories of the unforced flow and can lead to a satisfactory solution for the two-obstacle problem. The reader is referred to Pratt (1984) for more details.

**Figure Captions**

Figure 1. Long-wave solutions for hydraulically controlled flow over two consecutive obstacles of identical heights. Dashed curves show segments where the flow is vulnerable to a shock-forming instability.

Figure 2. Laboratory simulation of shallow flow over two obstacles of nearly the same height. The flow is from right to left. (From Pratt, 1984).

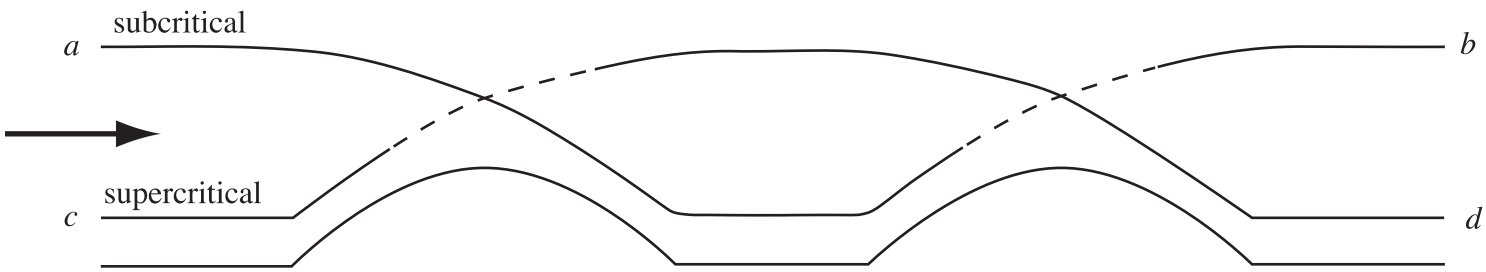


Figure 1.11.1


Figure 1.11.2 (low resolution)