Chapter 1: Review of the Hydraulics of Nonrotating Flows

Hydraulic behavior is closely related to wave propagation. This statement does not imply that waves need to be present; indeed, most 'text-book' examples involve steady flow. Rather, hydraulic behavior in a steady flow is related to the routing of information carried by waves through the fluid domain. Analyses of steady flows that appear throughout this book are often preceded or accompanied by discussions of linear and nonlinear waves. Linear wave dynamics are important in understanding the structure of steady flows and of regions of influence. Nonlinear wave dynamics are essential in the formation of hydraulic jumps and bores, and can be important in the establishment of hydraulically controlled steady states as the result of evolution from simple initial conditions. Some of our students skip through the material on waves and proceed directly to the discussions of steady flows; we encourage you to resist this temptation.

1.1 The long-wave approximation.

One of the elusive aspects of hydraulic phenomena is that nonlinearity is inherent to much of the important behavior. In order to make models that retain the nonlinear essence, but remain tractable, it is necessary to introduce a number of simplifying assumptions. Most are standard and will be familiar to readers having experience with geophysical fluid dynamics. For example, the flow is generally considered inviscid, nondiffusive, incompressible, and homogeneous (or at least stacked in homogeneous layers). However, the assumption that is most emblematic of hydraulics is that of gradual variations along the predominant direction of flow. Sometimes referred to as the *long wave* approximation, the assumption can be illustrated by considering a current confined to a channel aligned in the y-direction (Figure 1.1.1a). The channel guides the flow more or less along the y-*axis*. The fluid depth $d(x,y,t)$ and bottom elevation $h(x,y)$ may vary across the channel (Figure 1.1.1b). The y- and z- velocity components are given by $v(y,z,t)$ and $w(y,z,t)$. If the scale L of variations of d, v, and w in the y-direction is large compared to the scale of cross-sectional variations of these variables, then we say that the flow is gradually varying. The scale L might be associated with width variations (Figure 1.1a) or topographic variations (Figure 1.1.1c), or L might simply be the wavelength of a disturbance propagating along a uniform channel. Scales characterizing cross-sectional variations of the flow might include the mean depth and the channel width. In coastal geometries, rotating flows can be trapped to a coast within a distance set by internal dynamics. For example, currents may be trapped within the Rossby radius of deformation, to be defined later. Other oceanic and atmospheric flows that move freely of lateral boundaries have their own width scales and a definition of gradual variations may be made accordingly. In all cases, the long wave approximation is satisfied when the *largest transverse scale is much less than the smallest longitudinal scale*. The chief mathematical simplification gained is separation: the transverse (x- and z-) structure of the flow may be resolved, at least in a general form, before the y- or t-dependence need be considered.

Shallow-water theory for a homogeneous layer will serve as the basis for many of the examples that follow. The shallow water equations are themselves a product of the long-wave approximation. Consider their derivation for the case of a two-dimensional, inviscid, incompressible flow with a free surface. Begin with the Euler equations in two-dimensions:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{1.1.1}$$

$$\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g, \tag{1.1.2}$$

and

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{1.1.3}$$

Here ρ denotes the (uniform) density, p the pressure, and g the gravitational acceleration.

Subtraction of $\partial(1.1.1)/\partial z$ from $\partial(1.1.2)/\partial y$ and use of (1.1.3) lead to

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)=0, \tag{1.1.4}$$

showing that the vorticity $\partial w/\partial y-\partial v/\partial z$ is conserved following fluid elements.

The long-wave approximation for this flow is valid if variations in y, which occur over scale L, are gradual in relation to the depth scale D. The continuity equation (1.1.3) then suggests that $w/v=O(D/L)<<1$ and this leads to two simplifications involving the cross-sectional (z-) structure of the flow. First, the fluid vorticity is dominated by the factor $-\partial v/\partial z$. If the vorticity is zero, as would occur if motion has been conservatively generated from a state of rest, then $\partial v/\partial z=0$ for all space and time. The horizontal velocity therefore would remains independent of z:[1]

$$v=v(y,t). \tag{1.1.5}$$

The second simplification is that the pressure p becomes hydrostatically balanced:

$$\partial p/\partial z=-\rho g.$$

[1] More general structures are also possible. If the vorticity is initially finite but uniform then $v(y,z,t)=az+b(y,t)$.

This result can be deduced from (1.1.2) by showing that each term on the left-hand side becomes small compared to g when w/v=O(D/L)<<1 . If the bottom lies at $z=h(y)$ and the free surface at $z=h(y)+d(y,t)$, and the surface pressure is zero, then

$$p(y,z,t)=\rho g(d(y,t)+h(y)-z)\,. \qquad (1.1.6)$$

The transverse (z-) structure of the flow field has now been determined. The y-velocity is constant, the pressure varies linearly, and it is left as an exercise for the reader to show w also varies linearly. One may now proceed to the task of determining the y- and t- dependence of the variables. To do so, one must first develop the shallow water equations from the above expressions, a task left as an exercise (see #2 below) to the reader who has not seen this done. The general point is that that the cross-sectional structure of the flow can be determined at the outset, allowing one to concentrate on other aspects of the problem. We will return to this theme repeatedly as part of the consideration of flows with stratification or with structure in the cross-channel (x-) direction.

Exercises

1) Consider the stream function $\psi(x,y,t)$ for the flow described by (1.1.4), noting that

$$\frac{\partial\psi}{\partial y}=-w \qquad \text{and} \qquad \frac{\partial\psi}{\partial z}=v\,.$$

(a) When the flow is steady, show that

$$J_{y,z}(\psi,\partial w/\partial y-\partial v/\partial z)=0\,;$$

that is, the fluid vorticity is conserved along streamlines.

(b) Deduce from these relations the result

$$\nabla^2\psi=F(\psi)$$

where $F(\psi)$ specifies the value of the vorticity along each streamline ψ=const.

(c) Show that, in the long-wave limit, this last relation reduces to

$$\frac{\partial^2\psi}{\partial z^2}=F(\psi)\ .$$

(d) Suppose that at some upstream section $\psi=\alpha z^2/2$. Show that at any other section, the velocity v is described by $\alpha z+\beta(y)$. Note that our ability to write down a general form for the z-structure of the flow at an arbitrary location is made possible by the long wave approximation.

2. *Derivation of the shallow water equations*. This exercise makes use of the kinematic boundary conditions at the bottom and at the free surface of the fluid layer:

$$w(y,h,t) = v(y,t)\frac{\partial h}{\partial y} \qquad (1.1.7a)$$

$$w(y,h+d,t) = \frac{\partial d}{\partial t} + v(y,h+d,t)\frac{\partial (h+d)}{\partial y} \qquad (1.1.7b)$$

(a) Integrate the continuity equation (1.1.3) over the depth of the fluid and use (1.1.7) to obtain

$$\frac{\partial d}{\partial t} + \frac{\partial (vd)}{\partial x} = 0$$

(b) Substitute the expression for the pressure and velocity into 1.1.1 to obtain

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\left(\frac{\partial d}{\partial y} + \frac{\partial h}{\partial y}\right)$$

The results obtained in (a) and (b) are the shallow water equations.

Figure Captions

Figure 1.1.1 Definition sketch showing the channel geometry and the flow variables.

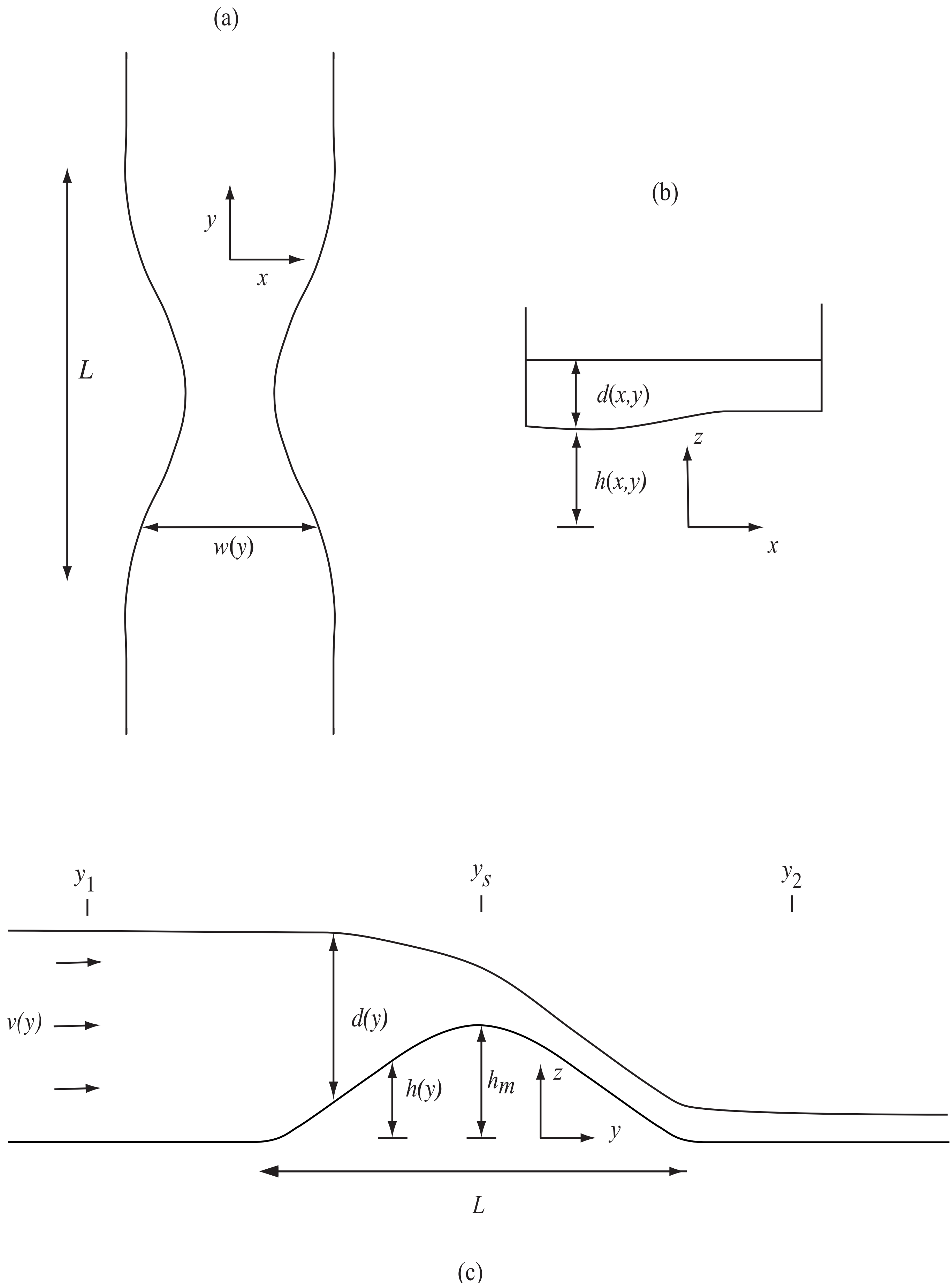

Fig. 1.1.1