1.3 Nonlinear steepening and rarefacation.

A basic knowledge of the hydraulic properties of a steady flow requires that one understand the characteristics of linear disturbances that propagate on that flow. However, some grasp of the elements of nonlinear propagation are crucial in understanding how hydraulic jumps and other types of shock waves are formed. This subject will also be of assistance when we explore the formation of steady solutions in laboratory or numerical experiments. A feature common to most nonlinear disturbances that arise in hydraulic models is that they are governed by hyperbolic partial differential equations. The defining characteristics of quasi-linear hyperbolic systems in 2 dimensions are described in detail in Appendix B, as are the methods for transforming the governing equations into standard forms. However, a heuristic definition would center on the properties that two independent types of disturbances (waves) exist and that these waves propagate through the physical domain at finite speeds. In the example of the previous section, the disturbances consisted of two linear gravity waves with speeds $c_{\pm} = V \pm (gD)^{1/2}$, propagating on a uniform background flow. As we now show, standard methodology also allows one to deal with wave amplitudes sufficiently large to destroy the distinction between the wave and the background flow.

To begin, it is helpful to rewrite the one-dimensional shallow water equations (1.2.1) and (1.2.2) in the form

$$\frac{d_{\pm}R_{\pm}}{dt} = -g\frac{dh}{dy} \tag{1.3.1}$$

where

$$\frac{d_{\pm}}{dt} = \frac{\partial}{\partial t} + [v \pm (gd)^{1/2}]\frac{\partial}{\partial y} \tag{1.3.2}$$

and

$$R_{\pm} = v \pm 2(gd)^{1/2}. \tag{1.3.3}$$

The procedure for obtaining this new form is discussed in Exercise 1 and the reader who seeks a more general discussion can consult Appendix B or look at standard texts such as Courant and Friedricks (1976) or Whitham (1974).

To interpret (1.3.1-1.3.3) first note that the operators $\frac{d_{\pm}}{dt}$ are time derivatives seen by observers traveling with *characteristic speeds*

$$\frac{dy_{\pm}}{dt} = v \pm (gd)^{1/2}. \tag{1.3.4}$$

These speeds are nothing more than the linear wave speeds with V and D replaced by the local velocity and depth, v and d. As before, it is helpful to think of the characteristic speeds as defining individual signals that move through the fluid and that compose general wave forms. Since the characteristic speeds vary throughout the flow field, different parts of a wave form move at different rates, leading to steepening (convergence) or rarefacation (spreading) of this form. If the bottom slope is zero ($dh/dy = 0$), an observer moving at one of the characteristic speeds sees a fixed value of the corresponding Riemann invariant R_+ or R_-. The latter are nonlinear generalizations of the functions introduced in the previous section. Among other things, they serve as indicators of the presence of 'forward' and 'backward' wave forms. If, for example, R_- is uniform in y, then the flow field contains no 'backward' wave forms (i.e., those propagating at speed v-$(gd)^{1/2}$). The forward propagating waves in such a field are sometime called *simple waves*. A simple physical interpretation of Riemann invariants in terms of energy or momentum has proved to be elusive, but perhaps the reader has a suggestion.

The characteristic speeds have real and unequal values for all flows in which the depth is non-zero, implying that (1.2.1) and (1.2.2) are hyperbolic. The importance of this property is that solutions to the initial-value problems can be constructed using the method of characteristics. Suppose that one is given the initial conditions $v(y,0)$ and $d(y,0)$ for all y and asked to compute the evolution of the flow for $t > 0$. The initial values of the Riemann invariants are given by $R_+ = v(y,0) + 2[gd(y,0)]^{1/2}$ and $R_- = v(y,0) - 2[gd(y,0)]^{1/2}$ and, provided that $dh/dy = 0$, these values are conserved along *characteristic curves* (or 'characteristics'), paths traced out in the (y,t)-plane by moving at the appropriate characteristic speed . Unlike the case for the linear waves considered in the previous section, the slopes of the characteristic curves are generally not constant. They depend on the local values of the velocity and depth within an evolving flow field.

We can now lay out a procedure for solving any initial value problem involving smooth initial conditions $v(y,0)$ and $d(y,0)$. As in the previous section, the solution is described in the characteristic (y,t) plane (Figure 1.3.1a). Let $y_+(y_o,t)$ and $y_-(y_1,t)$ represent the characteristic curves originating from $y = y_o$ and $y = y_1$ on the y-axis. The slopes of these curves are given by (1.3.4) and can be calculated at $t = 0$ from the initial conditions. For $t > 0$ the slopes depend on the solution itself and remain to be determined. Suppose for the moment that these slopes, and thus the curves $y_+(y_o,t)$ and $y_-(y_1,t)$ themselves are known, and that the curves intersect at point p. Then the velocity and depth at p can be computed from the values of the Riemann invariants that are carried along the curves. The value of R_+ is the value carried along the curve $y_+(y_o,t)$ and is equal to its value at $y = y_o$ and $t = 0$. The value of R_- at p is that specified at $y = y_1$ and $t = 0$. Once the two are known, the velocity and depth are computed from

$$v = \tfrac{1}{2}(R_+ + R_-)\,, \qquad (1.3.5)$$

$$d = [(R_+ - R_-)/4]^2 / g. \qquad (1.3.6)$$

More generally, the shapes of the characteristics are not known in advance and there is no immediate way of knowing the origin of the characteristic curves passing through p. In practice, this problem is dealt with by calculation of the initial slopes of the characteristics from the values of c_- and c_+ all along the y-axis. Straight-line approximations of the characteristic curves having these initial slopes are then projected forward a time increment Δt. A provisional solution is then computed at $t = \Delta t$ by carrying the initial values of R_+ and R_- forward along these curves. The characteristic speeds that follow from the provisional solution will generally be different than the initial estimates, implying that the characteristic curves are not straight. However, a correction can be made and the whole process repeated. Once satisfactory values of v and d have been found at $t = \Delta t$, the solution may be advanced further in time through reiteration.

The method will continue to work as long as the '+' curves (or the '-' curves) do not begin to intersect each other. Should the latter occur, as at q in Figure 1.3.1a, multiple values of R_+ (or R_-) would apply at the same point and the solution would be overdetermined. This situation is associated with the formation of shocks, meaning discontinuities in v and/or d, a circumstance to be explored later. Note that when the channel bottom contains topography, $R_\pm$ are no longer conserved and must be computed by integration of (1.3.1) along characteristic curves. In either case the curves may be interpreted as paths along which information travels.

Elementary examples of nonlinear evolution can be constructed through the consideration of a simple wave, as generated from an initial condition with uniform R_- or R_+. Consider the initial condition shown at the base of Figure 1.3.1a, with shallow water to the right and deeper water to the left. Suppose further that the shallower region has uniform depth d_o and is motionless ($v = 0$). Then choose the initial velocity to the left of the shallow region such that R_- is uniform. The value of R_- can be found by evaluating $v-2(gd)^{1/2}$ in the shallow, quiescent region, leading to $R_- = -2(gd_o)^{1/2}$. R_- must have this value for all y and therefore for all y and t reached by '-' characteristics, provided they do not intersect. An immediate consequence is that v and d become linked by the relation

$$v = 2(gd)^{1/2} - 2(gd_o)^{1/2},$$

which follows from the definition of R_-. The definition of R_+ then leads to

$$R_+ = 4(gd(y,t))^{1/2} - 2(gd_o)^{1/2},$$

and thus $d(y,t)$ itself is conserved along each '+' characteristic curve. Since both R_+ and d are conserved, v must also be conserved along each such curve and the characteristic speed must be constant and equal to its initial speed:

$$\begin{aligned} c_+ &= v(y,t) + (gd(y,t))^{1/2} \\ &= 3(gd(y,0))^{1/2} - 2(gd_o)^{1/2}. \end{aligned} \qquad (1.3.7)$$

The slope $1/c_+$ of each '+' curve is therefore constant, though different curves have different slopes. For the disturbance shown in Figure 1.3.1a, characteristics emanating from the deeper part of the disturbance are tilted more steeply than those emanating from the shallower portion. The tilt of each curve is an indication of how rapidly the signal corresponding to a particular part of the disturbance travels. The signal itself can be identified as a particular value of the depth d. Here the larger depths on the left propagate to the right more rapidly than shallower depths on the right. The slope $|\partial d / \partial y|$ of the free surface will therefore increase in what is called *nonlinear steepening*. We leave it as an exercise for the reader to show that a disturbance of the type shown in Figure 1.3.1b would spread or *rarefy*[1] (provided R_- remains uniform). In other words, the left-hand (shallower) d-values would propagate more slowly than those to the right.

The steepening wave form in the above example formally leads to a singularity corresponding to the intersection point q in Figure 1.3.1a. As more rapid signals overtake slower ones, the free surface slope increases without bound and eventually multiple values of d occupy the same y. The formation of a singularity is not proof of a real world catastrophe but rather an indication of breakdown in the shallow water approximation. This breakdown occurs when the horizontal length of the steepening wave becomes as small as the fluid depth. Beyond this point the steepening may or may not be arrested due to the intervention of non-hydrostatic effects or possibly other processes not captured in inviscid shallow water theory. If the length of the ultimate wave form or shock remains comparable to the depth, it is possible to represent it as a discontinuity in depth within a shallow water model and to approximate its amplitude and propagation speed. The ideas involved, collectively known as *shock joining theory* will be discussed in later sections.

A further illustration of the power of the method of characteristics is provided by a nonlinear version of the dam break problem explored in the previous section. We now consider the motion resulting from the destruction of a barrier separating a resting fluid of depth D from a region with no fluid (Figure 1.3.2). The initial conditions are

$$d(y,0) = \begin{cases} D \; (\mathrm{y} > 0) \\ 0 \; (\mathrm{y} < 0) \end{cases} \tag{1.3.8}$$

and

$$v(y,0) = 0. \tag{1.3.9}$$

The solution to this problem, as posed, is non-unique. Different results are obtained depending upon how one deals with the discontinuity in initial depth at $y = 0$. A reasonable way to resolve this difficulty is to replace this discontinuity with a smooth, but abrupt, transition over $0 < y < y_T$, as shown in the figure. One must specify the initial values of d and v within this short region and the corresponding characteristic speeds and Riemann invariants can be used to compute the evolution. Different specifications lead to different outcomes and this is the source of the non-uniqueness. The calculation of the evolution becomes quite simple if d and v are chosen such that either R_+ or R_- is uniform in the abrupt region and has the same value (either $2(gD)^{1/2}$ or $-2(gD)^{1/2}$) as in the region $y < 0$. Then one of the Riemann invariants will be initially uniform throughout the

[1] The term originates from gas dynamics and refers to the decreasing gas density that occurs when the wave form spreads.

fluid, allowing application of the simplifications described above for 'simple' waves. The limit $y_T \to 0$ may be taken later in order to approach the original step geometry.

Following this idea further, suppose that R_- is initially uniform in the transitional interval $0 < y < y_T$. Its value must therefore be $-2(gD)^{1/2}$ in order to match that in $y < 0$. It follows from the definition of R_- in the transition region that

$$v(y,0) - 2(gd(y,0))^{1/2} = -2(gD)^{1/2} \qquad (0<y<y_T).$$

However, $d < D$ in $0 < y < y_T$, implying that $v(y,0) < 0$. In other words, the fluid in the vicinity of the barrier will initially move *to the left* after the barrier is removed. Obviously, the assumption of uniform R_- is not one that leads to a physically realistic evolution. On the other hand, the choice R_+ = uniform leads to

$$v(y,0) + 2(gd(y,0))^{1/2} = 2(gD)^{1/2} \qquad (0 < y < y_T) \tag{1.3.10}$$

so that $v(y,0) > 0$ in the vicinity of $y = 0$, as expected.

It is now easy to make a sketch of the characteristic curves for all y and t, as is done at the top of Figure 1.3.2. Since R_+ is uniform, all of the '-' characteristics are straight. Their slope is determined by the initial value of the characteristic speed:

$$c_-(y,0) = \begin{cases} -(gD)^{1/2} & (y < 0) \\ v(y,0) - (gd(y,0))^{1/2} = 2(gD)^{1/2} - 3(gd(y,0))^{1/2} & (0 < y < y_T) \end{cases}.$$

Since $d(y,0)$ decreases monotonically from D to 0 as y increases from 0 to y_T, c_- increases from $-(gD)^{1/2}$ to $2(gD)^{1/2}$ over the transitional interval. The '-' characteristic curves originating from this interval therefore fan out as shown in Figure 1.3.2. Since d and v are constant along these curves, the developing flow consists of a rarefaction wave. The leading edge ($d = 0$) of this wave moves to the right at speed $2(gD)^{1/2}$ whereas the rear edge moves to the left at speed $(gD)^{1/2}$. The leading edge speed is also the fluid velocity at the leading edge. One of the fanning characteristic curves has $c_- = 0$ and therefore points directly upwards. In the limit $y_T \to 0$ this curve lies at the position ($y = 0$) of the barrier. Thus, the flow at $y = 0$ immediately becomes steady and critical after the barrier is removed. The flow at all other y approaches this same critical state as $t \to \infty$. The depth d_∞ and velocity v_∞ of this final state are determined by the condition of criticality [$v_\infty = (gd_\infty)^{1/2}$] and by the uniformity of $R_+ = v_\infty + 2(gd_\infty)^{1/2} = 2(gD)^{1/2}$, leading to $v_\infty = \tfrac{2}{3}(gD)^{1/2}$ and $d_\infty = (\tfrac{2}{3})^2 D$. The volume transport per unit width of channel is therefore given by

$$v_\infty d_\infty = \left(\frac{2}{3}\right)^3 g^{1/2} D^{3/2} \tag{1.3.11}$$

If the initial depth in $y > 0$ is finite then the advancing edge of the wave forms a shock. Calculation of the solution for this case requires knowledge of shock joining theory. A reader interested in the solution can consult Stoker (1957) for the full solution.

Exercises

1) *Derivation of Riemann Invariants.* Obtain the homogeneous form of (1.3.1) from the shallow water equations (1.2.1) and (1.2.2) by the following procedure:

a) Try to write the homogeneous versions of (1.2.1) and (1.2.2) in the *characteristic form*

$$\left(\frac{\partial}{\partial t}+c_{\pm}\frac{\partial}{\partial y}\right)v+\alpha_{\pm}(v,d)\left(\frac{\partial}{\partial t}+c_{\pm}\frac{\partial}{\partial y}\right)d=0$$

by multiplying (1.2.2) by a factor $\alpha_{\pm}(v,d)$, adding the result to (1.2.1), and calculating $\alpha_{\pm}(v,d)$ and $c_{\pm}(v,d)$ such that the above form is achieved.

b) Use this result to find the functions $R_{\pm}(v,d)$ satisfying $\left(\frac{\partial}{\partial t}+c_{\pm}\frac{\partial}{\partial y}\right)R_{\pm}=0$.

2) Linearize the Riemann invariants $R_{\pm}$ about a uniform background flow $v = V$ and $d = D$. How do the resulting expressions relate to the traveling wave functions $f_+(y-c_+t)$ and $f_-(y-c_-t)$ defined in Section 1.2?

3) Consider the initial condition $v = 0$ and

$$d(y,0)=\begin{matrix} d_o & (|y|> \mathrm{L}) \\ d_o+a(1-\frac{|y|}{L}) & (|y|\le \mathrm{L}) \end{matrix}.$$

Although this initial condition does not formally give a 'simple wave' solution, a simple-wave character emerges in parts of the domain after a finite time has elapsed. Use this behavior to discuss the qualitative features of the nonlinear evolution of this disturbance and compare it with the linear result (Exercise 2 of Section 1.2).

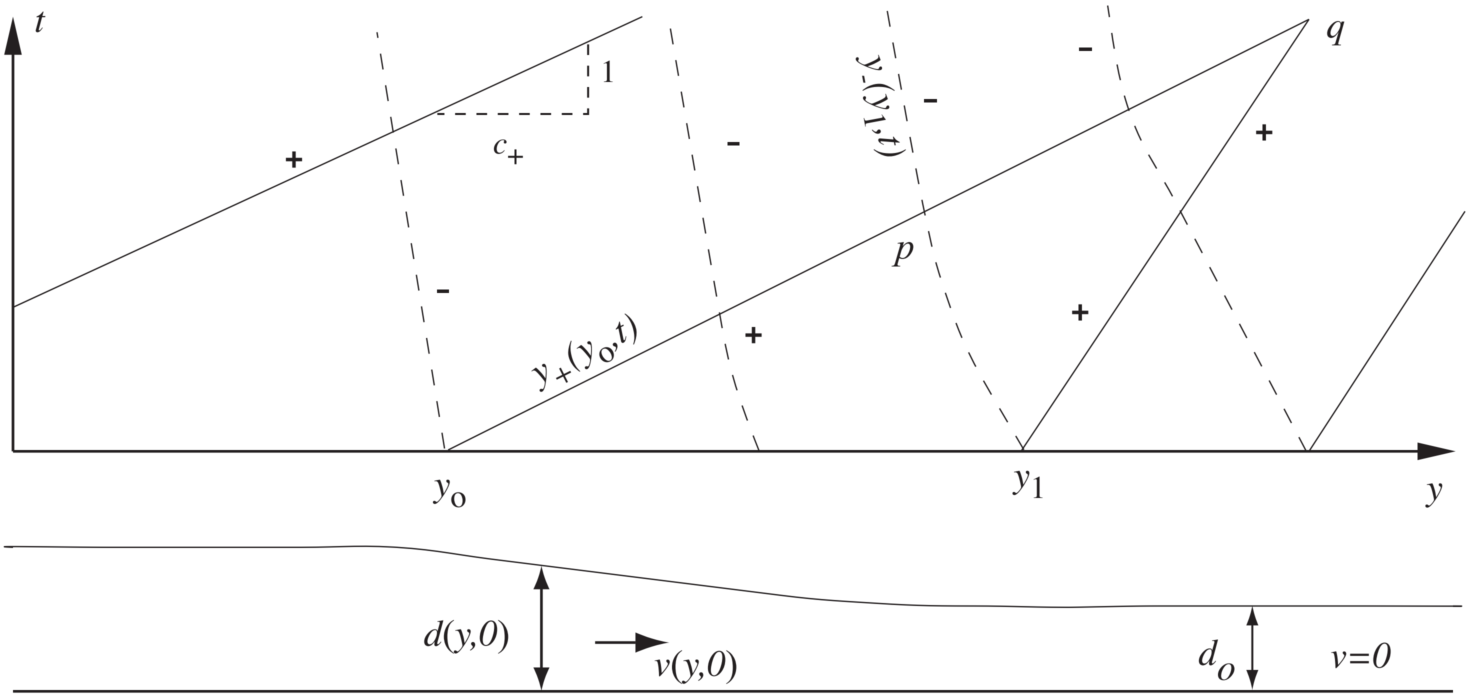
4) For the example shown in Figure 1.3.1a, at what time does wave breaking (shock formation) *first* occur? [Hint: do not necessarily be satisfied with the obvious answer.]

5) Consider the following twist on the classical dam break problem with initial conditions (1.3.8) and (1.3.9). Suppose that at $t = 0$ the barrier is not destroyed but instead is made to recede from the reservoir at a constant speed $c_o < 2(gD)^{1/2}$. Use the method of characteristics to sketch the solution.

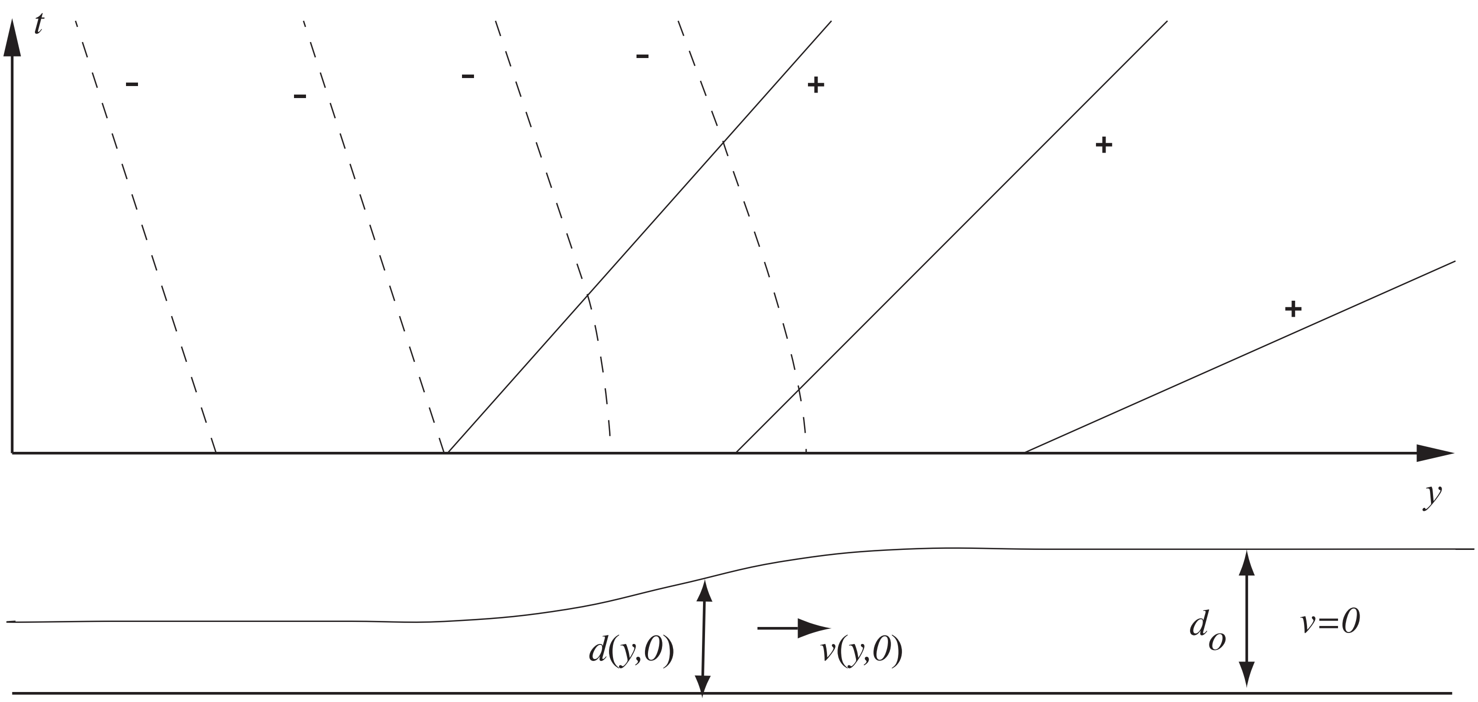
Figure Captions

1.3.1 Characteristic curves for two initial value problems, one with deeper water to the left (a) and the second with deeper water to the right (b). The solid and dashed curves represent 'plus' and 'minus' characteristic curves corresponding to $dy_{\pm} / dt = c_{\pm}$

1.3.2 The full dam break problem as visualized with a gradual initial change in depth, rather than a discontinuity, near $x = 0$. The characteristic curves are shown in the upper frame and the rarefying surface disturbance and intrusion are shown in the lower frame.

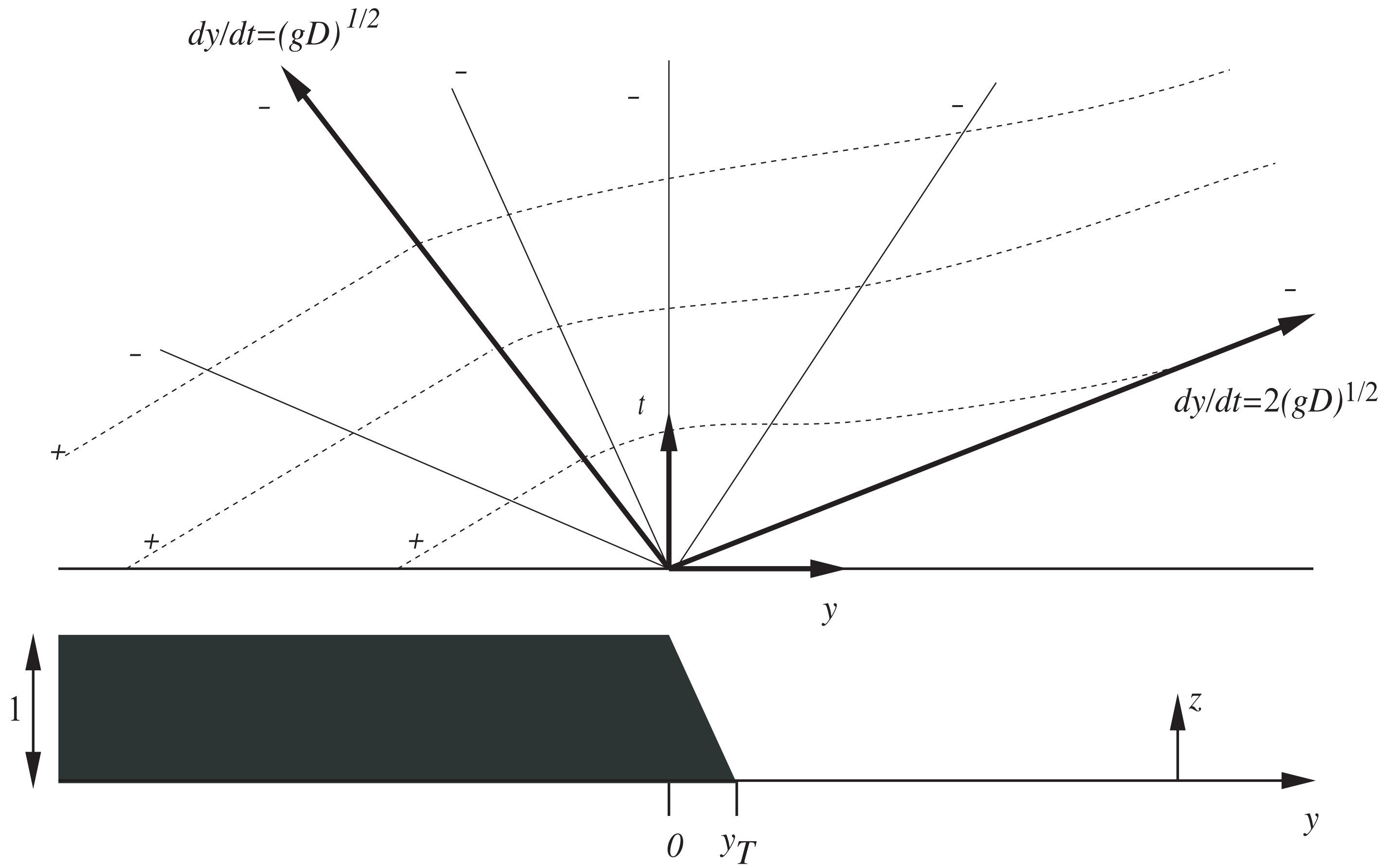


(a)



(b)

Fig 1.3.1



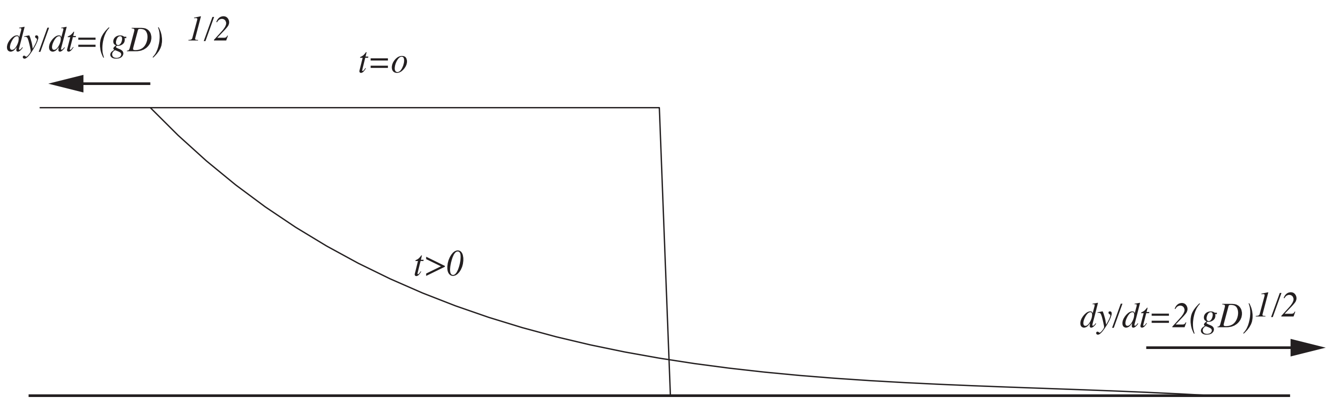


Figure 1.3.2