1.4 The hydraulics of steady, homogeneous flow over an obstacle.

We are now in a position to review one of the simplest examples of hydraulic behavior: that of a steady, homogeneous, free-surface flow passing over an obstacle or through a sidewall contraction. The channel will continue to have rectangular cross-section with gradually varying width *w* and bottom elevation *h*. The governing steady shallow-water equations are

$$v\frac{\mathrm{d}v}{\mathrm{d}y}+g\frac{\mathrm{d}d}{\mathrm{d}y}=-g\frac{\mathrm{d}h}{\mathrm{d}y} \qquad (1.4.1)$$

and

$$d\frac{\mathrm{d}v}{\mathrm{d}y}+v\frac{\mathrm{d}d}{\mathrm{d}y}=-vdw^{-1}\frac{\mathrm{d}w}{\mathrm{d}y}\,. \qquad (1.4.2)$$

Some of the general properties of the flow can be deduced by elimination of $\frac{\mathrm{d}v}{\mathrm{d}y}$ between the first two equations in favor of $\frac{\mathrm{d}d}{\mathrm{d}y}$. There follows

$$\frac{\mathrm{d}d}{\mathrm{d}y}=\frac{\frac{\mathrm{d}h}{\mathrm{d}y}-F_d^2\frac{d}{w}\frac{\mathrm{d}w}{\mathrm{d}y}}{F_d^2-1}, \qquad (1.4.3)$$

where $F_d^2=v^2/gd$. This expression gives the rate of change of the fluid depth along the channel in terms of *v* and *d* and in terms of the rate of change of the geometrical parameters *w* and *h*. Positive values of the numerator on the right hand side are associated with constrictions of the geometry due to increasing bottom elevation or to decreasing width. If the flow is subcritical ($F_d^2<1$), the denominator is negative and the fluid depth decreases in response to contractions. This is the situation when flow in a reservoir approaches a dam. Supercritical flow ($F_d^2>1$) experiences increases in depth in response to constrictions, a situation that can be observed in river rapids; where the water passes over a boulder, the depth increases and the free surface bulges out. Finally, critical flow ($F_d^2=1$) with a finite free-surface slope requires that the rate of contraction be zero: $\frac{\mathrm{d}h}{\mathrm{d}y}=\frac{d}{w}\frac{\mathrm{d}w}{\mathrm{d}y}$. This *regularity condition* holds where $\frac{\mathrm{d}h}{\mathrm{d}y}$ and $\frac{\mathrm{d}w}{\mathrm{d}y}$ are both zero, as at the crest or sill of an obstacle in a constant-width channel, at a narrows of a constant-elevation channel, or at a section where the minimum width coincides with a sill. Critical flow can also occur where increases in bottom elevation coincide with increases in width, or vice versa, such that the rate of geometrical contraction is zero according to the above criterion. Locations of critical flow are called *critical* or *control sections*.

Now consider the class of steady flows that arises when w is constant and the channel contains a single obstacle of height h_m, as shown in Figure 1.1.1c. Normally, computation of the flow is carried out using statements of conservation of energy:

$$\frac{v^2}{2} + gd + gh = B\,, \qquad (1.4.4)$$

and conservation of volume transport:

$$vdw = Q\,, \qquad (1.4.5)$$

obtained through the integration of (1.4.1) and (1.4.2) with respect to y. The constants B and Q represent the Bernoulli 'head' and volume flow rate. The former is the energy per unit mass of a fluid parcel and is always independent of depth in our slab-like, shallow-water system. For steady flow, (1.4.4) indicates that B is independent of y as well.

Solutions for the fluid depth can be found by eliminating v between (1.4.4) and (1.4.5), with the result:

$$\frac{Q^2}{2w^2d^2} + gd = B - gh. \qquad (1.4.6)$$

The quantity B-gh, sometimes called the *specific energy,* is the total energy minus the potential energy provided by the bottom elevation. It represents the intrinsic energy of the flow. Changing the bottom elevation alters the specific energy, forcing the depth to adjust to new values. One approach to the steady flow problem is to imagine that Q and B are predetermined, say, by conditions set far upstream of the obstacle, so that one can march along the channel, using (1.4.6) to calculate the depth at each point along the way. Of course (1.4.6) is cubic and there may be more than one value of d for each h, a situation which can be clarified by plotting h (or, more conveniently, B-gh) as a function of d. To make such a plot as general as possible, we first render (1.4.6) dimensionless by dividing by gD, where D is a scale chosen for convenience as $(Q/w)^{2/3}g^{-1/3}$. Then let $\tilde{d} = d/D$, $\tilde{h} = h/D$, and $\tilde{B} = B/gD$. The resulting relation

$$\frac{1}{2(\tilde{d})^2} + \tilde{d} = \tilde{B} - \tilde{h} \qquad (1.4.7)$$

is plotted in Figure 1.4.1a. To construct a solution at a particular y, first set the normalized energy $\tilde{B}$ and note the bottom elevation $\tilde{h}$ at that y. This fixes a point on the ordinate $\tilde{B} - \tilde{h}$. If the latter is > 3/2, two possible solutions for $\tilde{d}$ can be found. One corresponds to the left-hand branch and the other to the right-hand branch of the curve. There is one solution for $\tilde{B} - \tilde{h}$ = 3/2 corresponding to the minimum of the curve. Here

$$\frac{\partial}{\partial \tilde{d}}\left(\frac{1}{2\tilde{d}^2}+\tilde{d}\right)=-\frac{1}{\tilde{d}^3}+1=0\,. \qquad (1.4.8)$$

and therefore $\tilde{d}$ = 1 or $d = (Q/w)^{2/3}g^{-1/3} = (vd)^{2/3}g^{-1/3}$ or, finally, $F_d^2 = v^2/gd = 1$. The solution at the minimum of the curve therefore corresponds to critical flow. The left hand branch of the curve is associated with smaller depths and, since the flow rate is the same, larger velocities. Therefore the left-hand branch corresponds to supercritical ($F_d^2 > 1$) flow, while the right-hand branch corresponds to subcritical ($F_d^2 < 1$) flow. Constructing a solution requires choosing between the right- and left-hand branches, and there is nothing thus far to suggest how this choice is to be made.

Ignoring, for the moment, the dilemma of being forced to choose between two possible solutions, we arbitrarily begin on the subcritical branch of the solution curve. To construct a solution over a particular obstacle, begin at the section ($y = y_1$) upstream of the obstacle, where $\tilde{h}$ = 0. To find the depth $\tilde{d}$ at this section, go to Fig. 1.4.1a and read off the value $\tilde{d}(y_1)$ corresponding to $\tilde{B}-\tilde{h}(y_1) = \tilde{B}$. Next, move forward along the channel to where the bottom elevation $\tilde{h}$ begins to increase, causing $\tilde{B}-\tilde{h}$ to decrease. Remaining on the subcritical branch of the solution curve leads to lower values of $\tilde{d}$ as indicated by the arrows drawn above the curve. We can continue in this way until we reach the obstacle's sill at $y = y_s$ and $\tilde{h} = \tilde{h}_m$. If the sill elevation $\tilde{h}_m$ is sufficiently small that $\tilde{B}-\tilde{h}_m > 3/2$ the minimum of the solution curve is not reached and the depth $\tilde{d}(y_s)$ will exceed the critical depth $\tilde{d}$ = 1. Continuing further downstream causes one to retrace the solution curve in Figure 1.4.1a as indicated by the arrows drawn underneath. After the obstacle is passed ($y = y_2$), the depth returns to its upstream value. It is left as an exercise to show that where the depth decreases, the elevation $h + d$ of the free surface also decreases, so that the free surface will appear as shown in Figure 1.4.1b. Note that the solution is symmetrical in the sense that equal bottom elevations upstream and downstream of the sill see the same fluid depth. If the left-hand branch of the solution curve had been traced for the same topographic variations, a symmetrical supercritical solution with $\tilde{d}$ *increasing* over the obstacle would have resulted. We will refer to these solutions as pure subcritical or pure supercritical flow.

Next suppose that $\tilde{B}-\tilde{h}_m = 3/2$ so that the minimum of the solution curve is just reached at the sill. If the approach to the sill had been along the subcritical solution branch, there are two choices in constructing the downstream solution. First, one retraces the subcritical solution branch as in the above example. Second, one precedes onto the supercritical branch and thereby traces out an asymmetrical solution with the fluid depth *decreasing* in the downstream direction. This situation is depicted in Figure 1.4.2. As it turns out, the first of these scenarios results in a solution with a discontinuity in the free surface slope at the sill and can be ruled out. The proof of this result is the subject of Exercise 1 below. The preferred solution is thus the one with subcritical flow upstream, supercritical flow downstream, and critical flow at the sill. This type of flow, which resembles flow over a dam or spillway, is often described as being *hydraulically*

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controlled. The meaning of the term 'control' will soon become apparent. For now, we simply note that small disturbances generated downstream of the sill are unable to propagate upstream. The subcritical flow upstream of the sill is therefore immune to weak forcing imposed downstream of the sill.

It is also natural to ask what happens when $\tilde{B} - \tilde{h}_m < 3/2$, in which case no solution exists at the obstacle's crest. This situation occurs when the energy $\tilde{B}$ is insufficient to allow the fluid to surmount the obstacle. For example, one might start with the hydraulically controlled flow described above and raise the elevation of the sill a small amount, creating a small region about the sill for which no steady solution exists. Under these conditions a time-dependent adjustment must take place leading to a new upstream flow with a larger $\tilde{B}\ (= B(gQ/w)^{-2/3})$. This process is known as *upstream influence* and will be illustrated further in Sections 1.6 and 1.7. As we shall show, the value of $\tilde{B}$ has altered the minimal amount required to allow the flow to continue, implying that the new steady state is hydraulically controlled. Note that the change can be affected by increasing the Bernoulli function B or by decreasing the transport Q. Upstream influence over these quantities is an essential aspect of hydraulic control.

So far, we have constructed various solutions by fixing the energy parameter $\tilde{B}$ and varying the sill height of the topography. For a different perspective, consider the family of solutions obtained for a fixed topography by varying $\tilde{B}$. Figure 1.4.3 shows the free surface profiles of the solutions over an obstacle of unit dimensionless height. Each value of $\tilde{B}$ shown is associated with two solutions, one having supercritical and one subcritical flow upstream of the obstacle. For $\tilde{B} > 2.5$ the curves are the symmetrical, purely sub- or supercritical solutions discussed before. For $\tilde{B} = 2.5$ the two solutions intersect each other at the sill; one of these is the hydraulically controlled solution discussed above and the other, its mirror image, is supercritical upstream and subcritical downstream of the obstacle. For $\tilde{B} < 2.5$ the solutions are not continuous across the sill.

The asymmetrical solution that is supercritical upstream and subcritical downstream of the sill is unstable and probably unrealizable in most laboratory or field settings. A heuristic demonstration of the instability can be made through consideration of a small-amplitude disturbance imposed on the flow at the sill (Figure 1.4.4). This disturbance may be synthesized using the two linear wave modes of the system, which propagate at speeds $v-(gd)^{1/2}$ and $v+(gd)^{1/2}$. Since the slower wave propagates in the downstream direction *upstream* of the sill and in the upstream direction *downstream* of the sill, any energy carried in the slower mode will become focused and amplified at the sill. This situation will henceforth be referred to as the *shock forming instability*.

Some insight into the special requirements for the permissible location of critical flow can be gained through a consideration of the physics of resonance in a linear system. In general, an external force that translates along the channel with speed c_F tends to excite waves that have phase speed c_F. The efficiency of the excitation depends on how well the spatial structure of the disturbance projects on the free wave in question. In the steady, shallow flow under consideration, the forcing is due to the bottom topography (or width variations) and is clearly stationary. Therefore, resonant excitation can only occur when the wave in question is itself stationary; that is, the flow must be critical. Since the topography and flow have been constrained to vary gradually in the *y*-direction, the spatial structure of the forcing projects perfectly onto the wave. All of this implies that a steady critical flow cannot exist in the presence of forcing, a statement consistent with (1.4.3). Unless the forcing is zero (i.e. $\partial h/\partial y-(d/w)\partial w/\partial y=0$), the steady flow becomes singular. The reader might wonder why such behavior does not occur in connection with well-known stationary disturbances such as mountain lee waves. The answer is that such disturbances have finite wavelengths and are therefore non-hydrostatic. The significance of a lack of long-wave character is that the waves are dispersive, meaning that energy propagates at a different speed than phase. Thus the terrain may excite stationary waves, but radiation of energy away from the terrain limits local growth. Long gravity wave disturbances are characterized by equal phase and energy propagation (group) speeds.

If it is known in advance that the flow is hydraulically controlled, one can derive a transport or ‘weir’ relation that facilitates measurement of the volume transport. The goal is to monitor the discharge through measurements of the free surface elevation at some convenient location upstream of the control section. The procedure circumvents the need to directly measure the fluid velocity. Oceanographers would like to apply the same methodology to deep overflows, allowing them to avoid expensive and technically difficult velocity measurements. In such cases the deep flow is bounded above by an isopycnal (constant density) surface and the goal is to relate the deep transport to the elevation of this surface at some upstream location.

Derivations of weir formulas begin at the control section, where the flow is critical, and use conservation volume flux and energy in order to link conditions there to those at the monitoring location. As an example, consider a reservoir of width w_1 that drains across a sill of width w_s. The condition of criticality $v_c=(gd_c)^{1/2}$ at the sill can be used to write the volume transport $Q = vdw$ as $g^{1/2}d_c^{3/2}w_s$. Equating energy and volume transport at the reservoir $y = y_1$ and sill sections leads to

$$\frac{v_1^2}{2}+gd_1=\frac{v_c^2}{2}+gd_c+gh_m\,, \qquad (1.4.9)$$

and

$$v_1d_1w_1=v_cd_cw_s. \qquad (1.4.10)$$

Eliminating the velocities by combining these relations and using the critical condition leads to

$$\frac{3}{2}\left(\frac{gQ}{w_s}\right)^{2/3}-\frac{Q^2}{2d_1^2w_1^2}=g(d_1-h_m)=g\Delta z\,, \qquad (1.4.11)$$

where Δz is the difference in elevation between the free surface at $y = y_1$ and the sill. Measuring Δz and d_1 allows Q to be determined from the above formula. In many cases, y_1 can be chosen in a location where the depth d_1 or width w_1 is sufficiently large that the second term on the left-hand side can be neglected, resulting in the approximation

$$Q = \left(\tfrac{2}{3}\right)^{3/2} w_s g^{1/2} \Delta z^{3/2} . \qquad (1.4.12)$$

For the reduced gravity analog of the current model, weir formulas would permit the calculation of volume transport based on the interface elevation upstream of the critical section.

Although equation (1.4.11) was motivated by the practical necessity of measuring volume flux, it has a deeper meaning that bears on the concept of hydraulic control. In a controlled state, there is a fixed relationship between the parameters governing the flow, in this case Q and Δz, and the geometrical parameters describing the control section, in this case the sill height h_m. For non-controlled solutions no such relationship exists, implying that one has more freedom to manipulate these flows. We will elaborate on this point further. In addition, it is easy to show that critical flow is associated with a number of variational properties of steady flows. For fixed Q and h, the energy B of the flow is minimized over all possible values of d, which can be seen from Figure 1.4.3 or from taking $\partial B / \partial d = 0$ in (1.4.6). Similarly, it can be shown that for fixed B and h, Q is maximized over all d. Hydraulically controlled solutions thus minimize the energy available at a given volume transport, which is consistent with the idea that the fluid is barely able to surmount the obstacle. In addition, these solutions tend to maximize the transport available at a given energy level.

Exercises

1) Using l'Hôpital's rule in connection with (1.4.3), derive an expression for the slope of the free surface at a sill under critical flow conditions. You may assume that the channel width is constant. From the form of the result, show that critical flow can occur over a sill ($\mathrm{d}^2 h / \mathrm{d}y^2 < 0$) but not a trough ($\mathrm{d}^2 h / \mathrm{d}y^2 > 0$). Also show that a solution passing through a critical state at a sill generally cannot be subcritical (or supercritical) both upstream and downstream of the sill without incurring a discontinuity in the slope of the free surface.

2) Construct a nondimensional solution curve akin to that of Figures 1.4.1 or 1.4.2 for the case of a channel of constant h but variable w.

3) Consider a 100 m-deep reservoir that is drained by spillage over a dam of height 99 m. Both the dam and reservoir have width $w = 100$ meters. Approximate

(a) The volume flow rate from the reservoir.
(b) The depth and velocity at the sill of the dam.
(c) If you used an approximation to answer (a), estimate the error.

4) Suppose that the channel has a triangular cross-section. The width w at any z is given by

$$w(z) = 2\alpha(z - h(y))$$

where $h(y)$ is the elevation of the bottom apex. The along channel velocity v and surface elevation are independent of x.

(a) Taking h = constant, find the speed of long surface gravity waves in the channel.
(b) For steady flow, formulate a solution curve like that of Figure 1.4.1a or 1.4.2a showing how the fluid depth varies with h.
(c) Show that the condition obtained at the extrema of the curve is the critical condition found in part (a).
(d) Write down the weir formula for the case in which the fluid originates from an infinitely deep reservoir and spills over a sill.

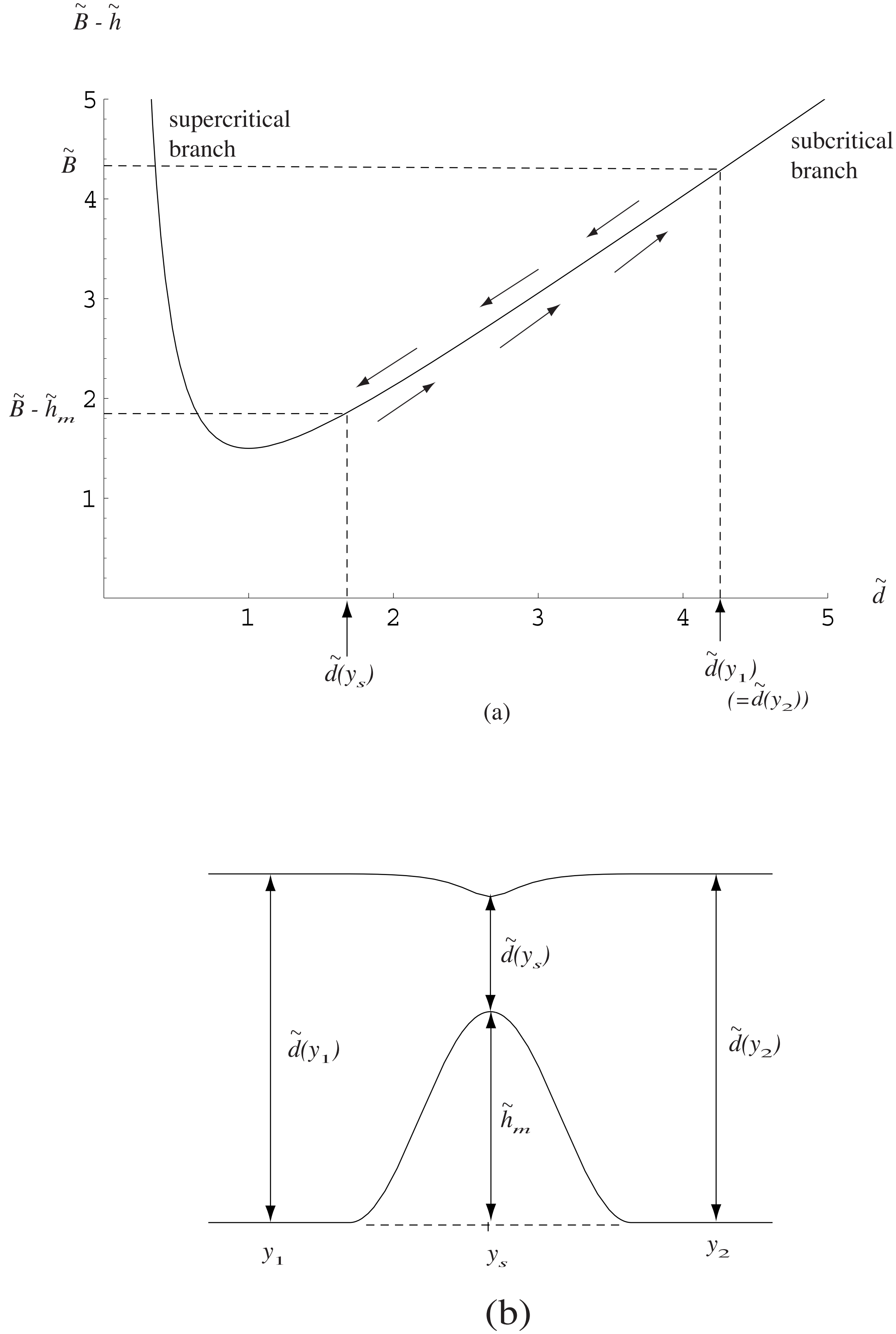
Figure Captions

1.4.1 (a) Plot of equation (1.4.1), with arrows indicating the route traced out by a subcritical solution. (b) Profile of a subcritical solution corresponding to the trace shown in (a).

1.4.2 Similar to Figure 1.4.1, but now showing a trace of a hydraulically controlled solution.

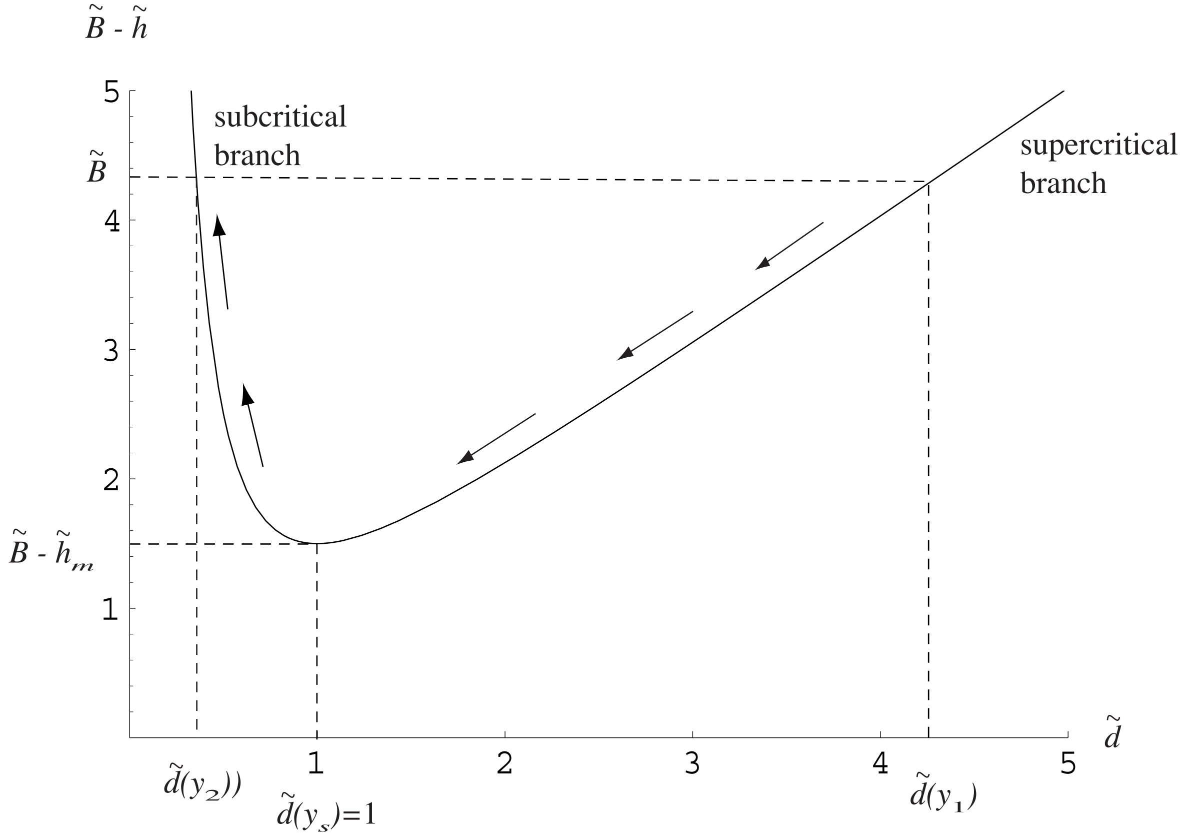
1.4.3 Free surface profiles for flow with different values of $\tilde{B}$ over the same obstacle.

1.4.4 Conditions leading to the shock forming instability. The flow supports two linear waves with speeds $v + (gd)^{1/2}$ and $v - (gd)^{1/2}$. In the supercritical-to-subcritical transition shown, the latter propagate towards the critical sill section from both upstream and downstream (wavy arrows).

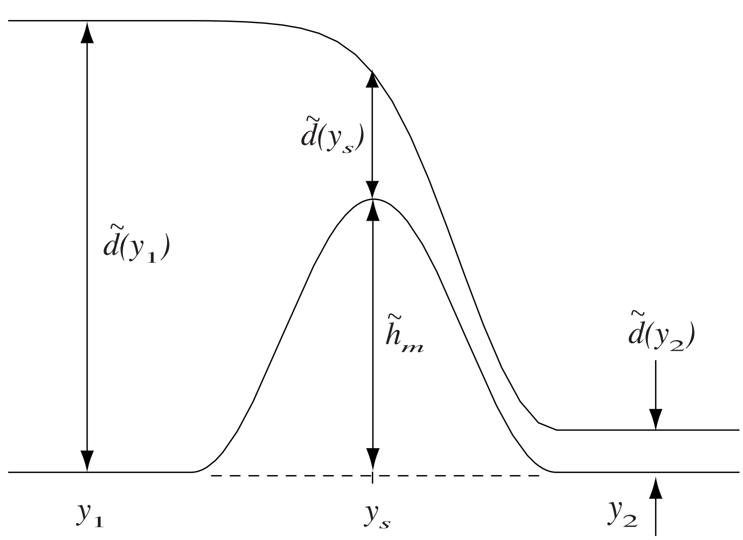


$\tilde{B}$ - $\tilde{h}$
5
4
3
2
1
$\tilde{B}$
$\tilde{B}$ - $\tilde{h}_m$
supercritical
branch
subcritical
branch
1
2
3
4
5
$\tilde{d}$
$\tilde{d}(y_s)$
$\tilde{d}(y_1)$
$(=\tilde{d}(y_2))$
(a)
$\tilde{d}(y_s)$
$\tilde{d}(y_1)$
$\tilde{d}(y_2)$
$\tilde{h}_m$
y_1
y_s
y_2
(b)

Figure 1.4.1

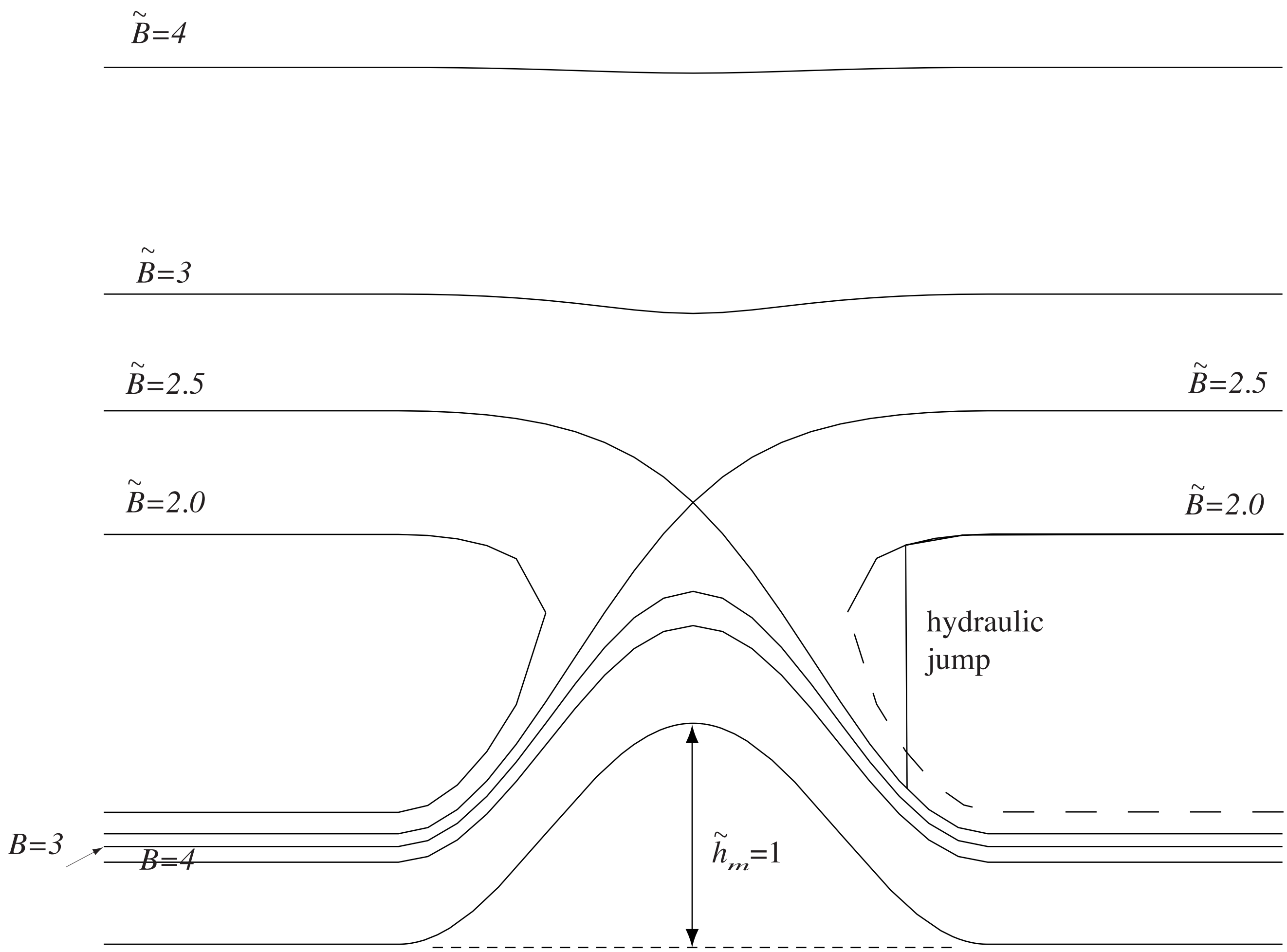


(a)



(b)

Figure 1.4.2



$\tilde{B}$=4
$\tilde{B}$=3
$\tilde{B}$=2.5
$\tilde{B}$=2.5
$\tilde{B}$=2.0
$\tilde{B}$=2.0
hydraulic
jump
B=3
B=4
$\tilde{h}_m$=1

Figure 1.4.3

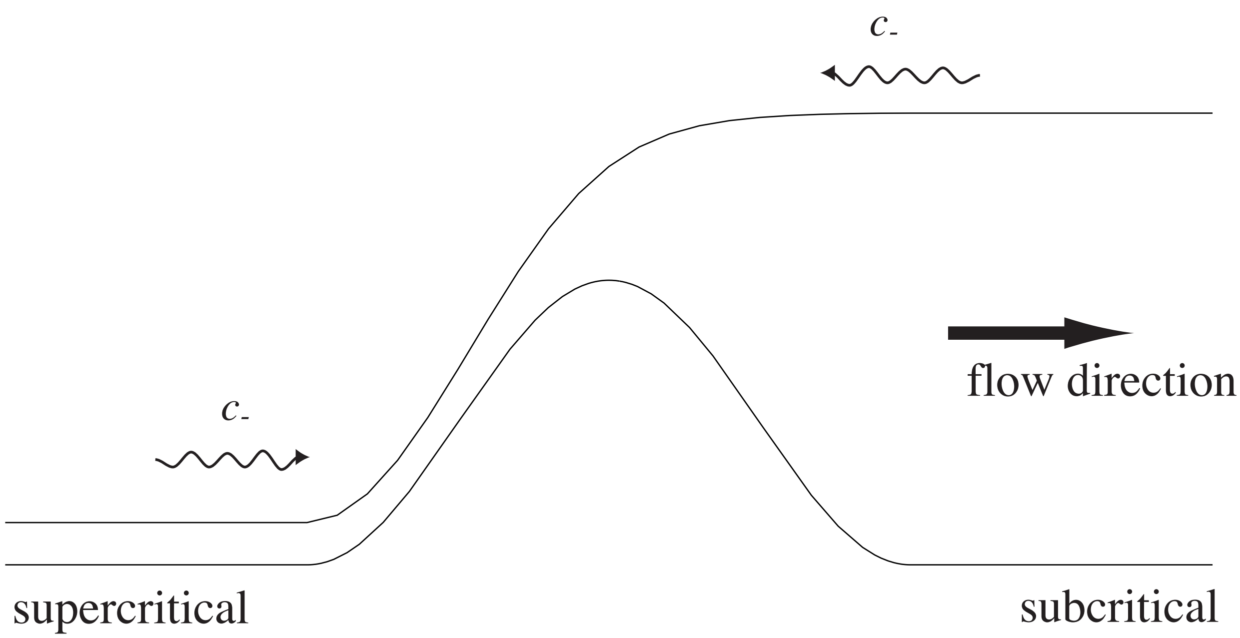


Figure 1.4.4