1.5 Hydraulics in Abstract.

In the example of the previous section, solutions in terms of the fluid depth d were obtained using conservation laws for the volume transport Q and the energy per unit mass B. For set values of these parameters, d depends only on the *local* bottom elevation h and channel width w, though several choices of d may be possible. There is no dependence on the values of h or w, or on the flow itself, at neighboring sections. As pointed out by Gill (1977), these elements are shared by a wide class of "hydraulic-type" systems. By taking advantage of the common elements, it is possible to develop machinery that allows a wide class of flows to be analyzed systematically.

The first hydraulic-type flow to be formally analyzed was probably the movement of a compressible gas through an orifice. The crucial result that the fluid velocity v in the orifice equals the speed of sound was derived independently by Reynolds (1886) and Hugoniot (1886). The statements of conservation of mass and energy are given by

$$\rho v A = M \text{ and } \frac{v^2}{2} + \int^{p} \frac{dp}{\rho} = B .$$

These are supplemented by an equation of state $\rho = F(p)$. Here A is the cross-sectional area of the conduit and M is the (constant) mass flux. The equations are analogous to our shallow-water model, with d playing the role of density ρ. Hugoniot was aware of experiments in which the velocity of the gas was observed to monotonically increase through an orifice, where A first decreases and then increases. This upstream/downstream asymmetry with respect to A is analogous to the asymmetry of d with respect to h and is characteristic of hydraulic transitions. Reynolds knew of an experiment in which the upstream propagation of information appeared to be blocked within the orifice:

"*Amongst the results of Mr. Wilde's experiments on the flow of gas, one, to which attention is particularly called, is that when gas is flowing from a discharging vessel through an orifice into a receiving vessel, the rate at which the pressure falls in the discharging vessel is independent of the pressure in the receiving vessel until this becomes greater than about five tenths the pressure in the discharging vessel.*"

The critical condition in the orifice was derived by both authors, essentially by considering the pressure decrease in a continuously narrowing conduit. They showed that a minimal possible pressure (Reynolds) or maximum possible ρv (Hugoniot) is reached when A is sufficiently small and they both observed that the implied v is equal to the speed of sound in the gas. (The details of the derivation are explored in Exercise 1.) The minimum in pressure found by Reynolds is analogous to the minimum in specific energy $\tilde{B} - \tilde{h}$ exhibited by the curves in Figures 1.4.1. The existence of a minimum or maximum implies that more than one v is possible for a given A, at least within a certain range of A. The minimization or maximization of properties as a way of obtaining a control criterion is sometimes referred to as a Hugoniot condition. The existence of more than one possible solution at a given cross section is characteristic of hydraulics problems in general.

(a) Gill's Original Approach

In the gas dynamics model and the shallow-water analogy, the state of the flow at any section of the channel can be specified in terms of a *single* dependent variable. This variable, which we denote γ, is related to the local geometry *h, w,* etc along with the parameters *Q, B,* etc. by a conservation law of the form[1]

$$\mathcal{G}(\gamma(y); h(y), w(y), \cdots; B, Q, \cdots) = const.^{2} \qquad (1.5.1)$$

B and *Q* could be the energy and flow rate, or they could represent other conserved properties of the system. The value of γ at a particular *y* determines all other attributes of the flow at that section. The position *y* does not appear explicitly. The constant on the right-hand side, which appears in Gill's original formulation, may be disposed by redefining $\mathcal{G}$. We may therefore take the constant to be zero with no loss of generality.

In the shallow water example of the Section 1.4, $\gamma(y)$ is the fluid depth *d.* In particular the Bernoulli equation (1.4.6) may be written as

$$\mathcal{G} = \frac{Q^2}{2\gamma^2 w^2} + g\gamma + gh - gB \,.$$

Other form of $\mathcal{G}$ could be written down by using variables like *v* instead of *d.*

A useful identity

$$\frac{d\mathcal{G}}{dy} = \frac{\partial \mathcal{G}}{\partial \gamma}\frac{d\gamma}{dy} + \frac{\partial \mathcal{G}}{\partial h}\frac{dh}{dy} + \frac{\partial \mathcal{G}}{\partial w}\frac{dw}{dy} + \cdots = 0 \,, \qquad (1.5.2)$$

is obtained by differentiation of (1.5.1). This result is often just the differential form of a momentum or continuity equation. The reader may wish to verify that application of (1.5.2) to the previous shallow-water example leads back to the *y*-momentum equation.

Now consider the conditions under which free, stationary, long waves of small amplitude exist. By 'long' we mean disturbances that vary gradually in the *y*-direction,

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[1] Perhaps out of modesty, Gill used the symbol *J* to represent the function in (1.5.1). To honor him, and to avoid confusion with the Jacobian operator, we use the symbol $\mathcal{G}$.

[2] The constant on the right-hand side, which appears in Gill's original formulation, may be disposed by redefining $\mathcal{G}$. We may therefore take the constant to be zero with no loss of generality.

just as the steady flow does. By 'free' we mean disturbances that occur spontaneously and are independent of any forcing mechanism such as bottom slope. When a steady flow becomes hydraulically critical at a particular section $y = y_c$, it can support a free, stationary disturbance at that section. In other words, the steady state can be locally altered without changing either the conduit geometry or the upstream conditions. The altered flow must therefore have the same volume flux, energy, etc. as the undisturbed flow. Let γ_c represent the undisturbed state at the critical section and let γ' represent the disturbance. Then the altered flow $\gamma_c+\gamma'$ must also satisfy (1.5.1):

$$\mathcal{G}(\gamma_c + \gamma'; h(y_c), w(y_c), \cdots; B, Q, \cdots) = const..$$

Taylor expansion of this relation leads to

$$\mathcal{G}(\gamma_c + \gamma'; h(y_c), w(y_c), \cdot\cdot) = \mathcal{G}(\gamma_c; h(y_c), w(y_c), \cdot\cdot) + \gamma' \left(\frac{\partial \mathcal{G}}{\partial \gamma}\right)_{\gamma=\gamma_c} + \cdots = const..$$

Since the undisturbed flow must satisfy (1.5.10) the first term on the right-hand side is zero. It follows that

$$\frac{\partial \mathcal{G}}{\partial \gamma} = 0 \qquad (1.5.3)$$

at the critical section. In plane words, criticality implies that the steady flow at a fixed location (fixed h, w, etc.) can be altered by an infinitesimal amount $\delta\gamma$ such at that (1.5.1) remains satisfied ($\mathcal{G}$ equals the same constant).

One of the important aspects of (1.5.3) is that it formally links the minimization (or maximization) used by Reynolds and Hugoniot. In the example of the previous section, (1.5.3) implies that

$$\frac{\partial \mathcal{G}}{\partial d} = \frac{\partial}{\partial d}\left[\frac{Q^2}{2d^2w^2} + gd + gh - gB\right] = \frac{\partial}{\partial d}\left[\frac{Q^2}{2d^2w^2} + gd\right] = 0$$

and thus 'specific energy' $Q^2/2d^2w^2 + gd$ is minimized when the flow is critical. Engineering texts often used this minimization as a basis for defining critical flow, even though the physical motivation is not always transparent.

The flow state at a particular section can be computed by solving (1.5.1) with the specific constant for the values of γ. In hydraulic applications, more than one root is possible; the cubic equation (1.4.7) admits as many as two roots for the depth of the shallow flow at each h. The two depths can be seen in Figure 1.4.3, where the Bernoulli constant $\tilde{B}$ may be regarded as C. The condition (1.5.3) implies that the roots coalesce, as occurs at the sill. All of the behavior described above is thus linked to the tendency of the hydraulic function $\mathcal{G}$ to admit multiple roots. It is important to note that this property will be lost when the shallow-water (or other) governing equations are linearized.

A further constraint implied by flow criticality follows from setting $\partial\mathcal{G} / \partial\gamma = 0$ in (1.5.2) leading to

$$\left(\frac{\partial\mathcal{G}}{\partial h}\frac{dh}{dy}+\frac{\partial\mathcal{G}}{\partial w}\frac{dw}{dy}+\cdots\right)_{y=y_c}=0\,. \tag{1.5.4}$$

This condition restricts the locations $y=y_c$ at which critical flow can occur. The locations at which critical flow actually occurs are sometimes called *control* sections. To obtain (1.5.4), it has been assumed that the flow remains smooth at $y=y_c$, so that $d\gamma/dy$ is finite there. Thus (1.5.4) is often referred to as a *regularity* condition. In fact, the satisfaction of (1.5.4) is equivalent in shallow water theory to the requirement that the numerator in (1.4.3) vanishes. It can readily be seen from that equation that the requirement is a necessary condition that the slope of the free surface remain bounded.

As in Figure 1.4.3, critical flow generally occurs at a section (or sections) $y = y_c$ marking the transition between states supporting wave propagation in different directions. Strictly speaking, the flow is able to support stationary disturbances only at y_c and not at points immediately upstream and downstream. The stationary disturbances are therefore possible in theory but are difficult to visualize in most applications. They should not be confused with stationary lee waves, which involve waves of finite length.

The purely local dependence of the functional $\mathcal{G}$ on y is a product of the conservative nature of the flow and of the gradually-varying geometry. When dissipation or rapid variations are present, the y-dependence generally becomes non-local. Such systems can still exhibit forms of hydraulic behavior. Examples are discussed in Exercise 4 of this section and in Section 3.8.

We have seen that critical flow can form at a maximum in h (a sill) and it is natural to ask whether the same is true of a minimum in h. Guidance comes from differentiating (1.5.1) twice with respect to y and applying the critical condition (1.5.3), leading to

$$\frac{\partial^2\mathcal{G}}{\partial\gamma^2}\left(\frac{d\gamma}{dy}\right)^2=-\frac{\partial}{\partial y}\left(\frac{\partial\mathcal{G}}{\partial h}\frac{dh}{dy}+\frac{\partial\mathcal{G}}{\partial w}\frac{dw}{dy}+\cdots\right).$$

In order for real values of $\frac{d\gamma}{dy}$ to exist at the critical section,

$\frac{\partial}{\partial y}\left(\frac{\partial\mathcal{G}}{\partial h}\frac{dh}{dy}+\frac{\partial\mathcal{G}}{\partial w}\frac{dw}{dy}+\cdots\right)$ and $\frac{\partial^2\mathcal{G}}{\partial\gamma^2}$ must have opposite signs. This condition generalizes the concepts of expansions and contractions in the channel geometry. In the example of the previous section, $\frac{\partial^2 \mathcal{G}}{\partial \gamma^2} = \frac{3Q^2}{d^4} > 0$,

whereas $\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{G}}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}y} + \frac{\partial \mathcal{G}}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}y} + \cdots\right) = g\frac{d^2h}{dy^2}$, so the bottom curvature d^2h/dy^2 must be < 0. Negative curvature is characteristic of a sill but not a depression in the bottom and the implication is that physically meaningful critical solutions require a sill geometry. At the sill, $\left(\frac{d\gamma}{dy}\right)^2 = \frac{gd^4}{3Q^2}\frac{d^2h}{dy^2}$, indicating two possible free surface slopes. The two slopes are simply those of the intersecting solutions (both with $\tilde{B} = 2.5$) shown in Figure 1.4.3. Computation of a continuous solution through a critical section therefore requires a *hydraulic transition* in which subcritical upstream flow joins with supercritical downstream flow (or vice versa). One may not move through the critical point and remain on the subcritical branches.

b) Multiple Variables

Reduction of the problem to the single-variable format envisioned by Gill (1977) is not always easy. It is often more convenient, and sometimes necessary, to work with two independent relations in two variables γ_1 and γ_2:

$$\mathcal{G}_1(\gamma_1, \gamma_2; h, w, \cdots; B, Q, \cdots) = C_1 \tag{1.5.5}$$

and

$$\mathcal{G}_2(\gamma_1, \gamma_2; h, w, \cdots; B, Q, \cdots) = C_2 . \tag{1.5.6}$$

The approach to dealing with this system is laid out by Pratt and Armi (1988) and Dalziel (1991) and the generalization to an arbitrary number of variables is discussed by Lane-Serff *et al.* (2000) and Pratt and Helfrich (2005).

For the system (1.5.5 and 1.5.6), the existence of a stationary wave requires that small perturbations γ_1' and γ_2' of the flow exist at a fixed location such that the new altered flow remains a solution. Taylor expansion of $\mathcal{G}_1$ and $\mathcal{G}_2$ for fixed h, w, etc. about the unperturbed state leads to

$$\frac{\partial \mathcal{G}_1}{\partial \gamma_1}\gamma_1' + \frac{\partial \mathcal{G}_1}{\partial \gamma_2}\gamma_2' = 0 \tag{1.5.7}$$

$$\frac{\partial \mathcal{G}_2}{\partial \gamma_1}\gamma_1' + \frac{\partial \mathcal{G}_2}{\partial \gamma_2}\gamma_2' = 0 . \tag{1.5.8}$$

The critical condition is just the solvability condition for this pair:

$$\frac{\partial \mathcal{G}_1}{\partial \gamma_1}\frac{\partial \mathcal{G}_2}{\partial \gamma_2} - \frac{\partial \mathcal{G}_1}{\partial \gamma_2}\frac{\partial \mathcal{G}_2}{\partial \gamma_1} = 0\,. \qquad (1.5.9)$$

Stationary waves then involve the displacement ($d\gamma_1$, $d\gamma_2$) as given by (1.5.7) or (1.5.8):

$$\vec{\gamma}' = \gamma_1{}'\left(1, -\left[\frac{\partial \mathcal{G}_1 / \partial \gamma_1}{\partial \mathcal{G}_2 / \partial \gamma_2}\right]_{y=y_c}\right), \qquad (1.5.10)$$

where $d\gamma_1$ is small but arbitrary. The displacement vector contains information about the structure of the stationary wave (see Exercise 3).

The generalization of the regularity condition (2.5) can be found by writing out the identities $d\mathcal{G}_1/dy = 0$ and $d\mathcal{G}_2/dy = 0$:

$$\frac{d\mathcal{G}_1}{dy} = \frac{\partial \mathcal{G}_1}{\partial \gamma_1}\frac{d\gamma_1}{dy} + \frac{\partial \mathcal{G}_1}{\partial \gamma_2}\frac{d\gamma_2}{dy} + \frac{\partial \mathcal{G}_1}{\partial h}\frac{dh}{dy} + \frac{\partial \mathcal{G}_1}{\partial w}\frac{dw}{dy} + \cdots = 0$$

$$\frac{d\mathcal{G}_2}{dy} = \frac{\partial \mathcal{G}_2}{\partial \gamma_1}\frac{d\gamma_1}{dy} + \frac{\partial \mathcal{G}_2}{\partial \gamma_2}\frac{d\gamma_2}{dy} + \frac{\partial \mathcal{G}_2}{\partial h}\frac{dh}{dy} + \frac{\partial \mathcal{G}_2}{\partial w}\frac{dw}{dy} + \cdots = 0$$

Solving for $d\gamma_1 / dy$ leads to

$$\frac{d\gamma_1}{dy} = \frac{\dfrac{\partial \mathcal{G}_1}{\partial \gamma_2}\left(\dfrac{\partial \mathcal{G}_2}{\partial y}\right)_{\gamma_1,\gamma_2} - \dfrac{\partial \mathcal{G}_2}{\partial \gamma_2}\left(\dfrac{\partial \mathcal{G}_1}{\partial y}\right)_{\gamma_1,\gamma_2}}{\dfrac{\partial \mathcal{G}_1}{\partial \gamma_1}\dfrac{\partial \mathcal{G}_2}{\partial \gamma_2} - \dfrac{\partial \mathcal{G}_1}{\partial \gamma_2}\dfrac{\partial \mathcal{G}_2}{\partial \gamma_1}}$$

where $\left(\dfrac{\partial(\,)}{\partial y}\right)_{\gamma_1,\gamma_2} = \dfrac{\partial(\,)}{\partial w}\dfrac{\partial w}{\partial y} + \dfrac{\partial(\,)}{\partial h}\dfrac{\partial h}{\partial y} + \cdots$ is a derivative taken with γ_1 and γ_2 held constant. Critical flow requires that the denominator vanish and the numerator must then vanish if the flow is to remain well behaved. The regularity condition is thus

$$\frac{\partial \mathcal{G}_1}{\partial \gamma_i}\left(\frac{\partial \mathcal{G}_2}{\partial y}\right)_{\gamma_1,\gamma_2} - \frac{\partial \mathcal{G}_2}{\partial \gamma_i}\left(\frac{\partial \mathcal{G}_1}{\partial y}\right)_{\gamma_1,\gamma_2} = 0 \quad (i{=}1 \textit{ or } i{=}2)\,, \qquad (1.5.11)$$

(The $i = 2$ version, which follows from developing an expression for $d\gamma_2 / dy$, is not independent of the $i = 1$ version.)

The machinery is easily extended to problems governed by *N* relations for *N* independent variables:

$$
\begin{aligned}
&\mathcal{G}_1(\gamma_1(y),\gamma_2(y),\gamma_3(y),\cdots,\gamma_N(y);h(y),w(y),\cdots;B,Q,\cdots)=C_1\\
&\mathcal{G}_2(\gamma_1(y),\gamma_2(y),\gamma_3(y),\cdots,\gamma_N(y);h(y),w(y),\cdots;B,Q,\cdots)=C_2\\
&\qquad\qquad\qquad\qquad\vdots\\
&\mathcal{G}_N(\gamma_1(y),\gamma_2(y),\gamma_3(y),\cdots,\gamma_N(y);h(y),w(y),\cdots;B,Q,\cdots)=C_N\,.
\end{aligned}
\tag{1.5.12}
$$

The condition for stationary waves is now

$$\sum_{j=1}^{N}\frac{\partial\mathcal{G}_i}{\partial\gamma_j}\gamma_j{}'=0 \qquad (i=1,\,2,\,\cdots N), \tag{1.5.13}$$

and the corresponding solvability condition is the vanishing of the generalized Jacobian:

$$\det\left(\frac{\partial\mathcal{G}_i}{\partial\gamma_j}\right)^T=0\,, \tag{1.5.14}$$

where

$$\left(\frac{\partial\mathcal{G}_i}{\partial\gamma_j}\right)^T=\begin{pmatrix}\partial\mathcal{G}_1/\partial\gamma_1 & \cdot\quad\cdot & \partial\mathcal{G}_1/\partial\gamma_N\\ \cdot & \cdot\quad\cdot & \cdot\\ \cdot & \cdot\quad\cdot & \cdot\\ \partial\mathcal{G}_N/\partial\gamma_1 & \cdot\quad\cdot & \partial\mathcal{G}_N/\partial\gamma_N\end{pmatrix}. \tag{1.5.15}$$

The tangent displacement vector $(d\gamma_1,d\gamma_2,...)_c$, which is computed from any member of (1.5.13), again determines the transverse structure of the stationary wave.

It can also be shown (see Exercise 6) that the generalized regularity condition is

$$\det\left(\left(\frac{\partial\mathcal{G}_i}{\partial\gamma_j}\right)^T\middle|\left(\frac{\partial\mathcal{G}_i}{\partial y}\right)_\gamma^T\right)=0 \tag{1.5.16}$$

where $\left(\frac{\partial\mathcal{G}_i}{\partial\gamma_j}\right)^T\middle|\left(\frac{\partial\mathcal{G}_i}{\partial y}\right)_\gamma^T$ is the matrix obtained by replacing the *i*th column of $\left(\frac{\partial G_i}{\partial\gamma_j}\right)^T$ by

$$\left(\frac{\partial \mathcal{G}_i}{\partial y}\right)_\gamma^T = \begin{pmatrix} (\partial \mathcal{G}_1 / \partial y)_\gamma \\ (\partial \mathcal{G}_2 / \partial y)_\gamma \\ \vdots \\ (\partial \mathcal{G}_N / \partial y)_\gamma \end{pmatrix}.$$

When formulating hydraulic functionals $\mathcal{G}_1$, $\mathcal{G}_2$, etc. for a particular system, there is a disadvantage in reducing the system to a single functional in a single unknown. Namely, certain kinds of critical states may be missed in the evaluation of the critical condition (1.5.3) for the single-variable formulation. This difficulty arises when the stationary wave in question has no displacement in terms of the chosen single variable. That is, the tangent displacement vector $(\gamma_1', \gamma_2', \cdots)$ for a particular stationary disturbance may have a zero constituent, say γ_2'. If the system is reduced such that γ_2 is the only variable, then the critical condition for this disturbance will not be identified by (1.5.3). The missing critical condition will, however, be identified by the multivariate formula (1.5.14). An example will be given in Section 2.4.

Exercises

1) *Transonic flow in an isotropic gas*. Consider an inviscid and diffusion-free, compressible gas whose motion is governed by the following equations:

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \chi$$
$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0$$
$$p = \rho R T$$
$$\rho c_v \frac{dT}{dt} + p \nabla \cdot \mathbf{u} = 0$$

where T is the absolute temperature, c_v is the specific heat at constant volume, and χ is the body force per unit mass.

(a) Show that $\frac{d}{dt}\left(\frac{p}{\rho^{\hat{\gamma}}}\right) = 0$, where $\frac{c_v}{R} = \frac{1}{\hat{\gamma}-1}$. [*Hint: one starting point is elimination of* $\nabla \cdot$ $\mathbf{u}$ *from the second and fourth equations*.]

(b) Next consider the generalized form of the Bernoulli function for steady compressible flow:

$$\int \frac{dp}{\rho} + \Omega + \frac{|\mathbf{u}|^2}{2} = \text{constant along streamlines}$$

where Ω is the body force potential. Applying this and the steady form of the result in (a) to a one-dimensional flow in a wind-tunnel of slowly varying cross-sectional area A(y), derive a hydraulic functional of the form $\mathcal{G}(\rho; A) = C$. (The body force potential may be neglected.)

(c) From the result of (b), obtain a critical condition and deduce that the intrinsic signal speed (here the speed of sound) is $\left[\hat{\gamma} p / \rho\right]^{1/2}$.

2. *Homogeneous, free-surface flow with shear*. Following Garrett and Gerdes (2003) consider a steady, shallow, homogeneous flow with vertical shear ($\partial v / \partial z = 0$). The flow is described by a stream function $\psi(y,z)$ such that $\partial \psi / \partial z = v$, $\psi(y, h(y)) = 0$, and $\psi(y, h(y) + d(y)) = Q$. The Bernoulli function is given by

$$B(\psi) = \frac{v^2}{2} + gd + gh \,.$$

Construct a hydraulic functional for the flow by following these steps:

(a) Show that $d = \int_0^Q \frac{d\psi}{v}$ and therefore

$$d = \frac{1}{\sqrt{2}} \int_0^Q \frac{d\psi}{\left[B(\psi) - g(d+h)\right]^{1/2}} \,.$$

(b) Define a hydraulic functional $\mathcal{G}_2(d(y), v\ (y); h(y), w(y), \cdots; B, Q, \cdots) = 0$ based on the above relation. Show that setting $\partial \mathcal{G} / \partial d = 0$ leads to the critical condition:

$$g \int_h^{h+d} \frac{dz}{v^2} = 1$$

and compare this with the result of Exercise 4 of section 1.2.

3. Cast the hydraulic problem for homogeneous, free-surface flow in terms of two functionals $\mathcal{G}_1(d(y), v(y); h(y), w(y)) = Q$ and $\mathcal{G}_2(d(y), v(y); h(y), w(y)) = B$ representing the continuity and Bernoulli equations. Show that the critical and regularity conditions obtained using the two-variable (d,v) machinery is the same if the single-variable representation were used. Using (1.5.10), show that the displacement vector specifies a

relationship between the depth and velocity perturbations, and that this relationship is the same as that implied by (1.2.5) and (1.2.6) for the '-' wave.

4) *Non-local dependence on y*. Consider the functional

$$G_P(d(y),\int_{y_o}^{y} f(d)dy';h(y);q,y_o\cdots)=\frac{Q^2}{2d^2w^2}+d+h+\alpha\frac{Q^2}{w^2}\int_{y_o}^{y} d^{-3}dy'=B(y_o) \qquad \text{(A.1)}$$

governing a shallow flow under the influence of bottom drag (Pratt, 1986). Fixed parameters include the drag parameter α. The presence of drag introduces an integration from an upstream location $y = y_o$ where the depth and velocity v are known, to the section under consideration. Consider the possibility that a free, small amplitude, stationary disturbance exists at a section at $y = y_c$ but at no other upstream point. Show that a necessary condition for existence is

$$\lim_{\delta d\to 0}\left[\frac{G_P\left(d+\delta d,\int_{y_o}^{y_c} f(d+\delta d)dy';h(y_c),q,y_o\cdots\right)-G_P\left(d,\int_{y_o}^{y_c} f(d)dy';h(y_c),q,y_o\cdots\right)}{\delta d}\right]=0\,,$$

Show that evaluation of this limit leads to the critical condition $v = d^{1/2}$.

5) Suppose that the dimensionless obstacle height in (1.4.7) $\tilde{h}(y)=a\hat{h}(y)$, where $a << 1$. Let

$$\tilde{d}=\tilde{d}^{(0)}+a\tilde{d}^{(1)}+O(a^2)$$

where $\tilde{d}^{(0)}$ is the solution for $\tilde{h}$ = 0. Formulate a Gill function for the variable $\tilde{d}^{(1)}$, but show that there can be no hydraulic transitions. Why does linearization of the problem prevent this phenomenon?

6) By taking the y-derivatives of (1.5.12) and applying the critical condition, show, using Cramer's rule, that

$$\frac{d\gamma_i}{dy} = -\frac{\det\left(\left(\frac{\partial \mathcal{G}_i}{\partial \gamma_j}\right)^T \middle| \left(\frac{\partial \mathcal{G}_i}{\partial y}\right)_\gamma^T\right)}{\det\left(\frac{\partial \mathcal{G}_i}{\partial \gamma_j}\right)^T}.$$

Deduce the regularity condition (1.5.16).