1.7 Solution to the Initial-Value Problem

The shock joining relations developed in the previous section make it possible to solve the initial-value problem posed by Long's experiment. The term 'solve' is used advisedly here for we do not actually calculate the evolving flow during its early development. Instead, we wait until the various transients have separated from one another, at which point the flow field consists of steady segments separated by isolated bores and rarefaction waves. The formal solution is thereby guided by the experiment. Piecing together the different steady segments of flow permits a solution to be constructed and, more importantly, allows the calculation of the obstacle heights required to initiate partial or total blockage or establishment of a hydraulic jump.

Let us continue to view the problem as the adjustment to the sudden introduction of a stationary obstacle into a uniform stream. As noted in the previous section, permanent upstream effects (partial blockage) occur when the obstacle is sufficiently high that the initial flow has insufficient energy to ascend the crest or sill, at least according to a steady-state calculation. The critical obstacle height h_c is given by (1.6.1). Figure 1.7.1a shows the developing upstream flow for $h_m>h_c$. The flow state (v_0,d_0) far upstream, also the initial flow, is approached by a bore that moves at speed c_1 and establishes a new upstream state (v_a,d_a). Equations (1.6.4) and (1.6.5) can be used to link the two steady flows across the bore, leading to

$$(v_0-c_1)d_0=(v_a-c_1)d_a \qquad (1.7.1)$$

and

$$(v_0-c_1)^2d_0+gd_0^2/2=(v_a-c_1)^2d_a+gd_a^2/2\,. \qquad (1.7.2)$$

In addition, conservation of energy and mass connect the sill flow with the steady flow immediately upstream of the obstacle according to

$$v_ad_a=v_cd_c \qquad (1.7.3)$$

and

$$\frac{v_a^2}{2}+gd_a=\frac{v_c^2}{2}+g(d_c+h_m)\,. \qquad (1.7.4)$$

Adding the condition that the sill flow is critical,

$$v_c=(gd_c)^{1/2}\,, \qquad (1.7.5)$$

results in five equations for the unknowns c_1, d_a, v_a, v_c, and d_c.

The locations of the different solution regimes can be plotted (Figure 1.7.2) in terms of the dimensionless obstacle height h_m/d_0 and initial Froude number F_0. The curve BAE gives the critical obstacle height h_c/d_0 in terms of F_0 and is determined by (1.6.1). To the left of this curve the obstacle is shorter than the critical height and the steady flow established is completely supercritical or subcritical, depending on the initial Froude number. No upstream influence exists. To the right of this curve, upstream (and downstream) influence occurs and the flow adjusts to a hydraulically controlled steady state. As we have shown, the upstream influence takes the form of a bore that partially blocks the flow. Note that *any* bore that propagates upstream must decrease the volume transport, a property that can be deduced from conservation of mass (1.7.1) in the form:

$$v_a d_a = v_0 d_0 + c_1 (d_a - d_0)\,. \qquad (1.7.6)$$

Since $c_1<0$ and $d_a>d_o$ the final transport is less than the initial transport ($v_a d_a < v_0 d_0$) and we say that the flow is partially blocked. Various properties of the solution including the bore speed and the final transport can be obtained by solving (1.7.1-1.7.5) and some of these properties are presented in Baines (1995, figures 2.10 and 2.12).

Further to the right in the diagram curve BC gives the value of h_m/d_0 needed to completely block the flow. The governing relation (see Exercise 1) is given by

$$F_0 = \left(\frac{h_m}{d_0} - 1\right)\left(\frac{1 + h_m / d_0}{2 h_m / d_0}\right)^{1/2}. \qquad (1.7.7)$$

The wedge shaped region EAF in Figure 1.7.2 represents special initial conditions for which two final steady states are possible, depending on how the experiment is carried out. Consider the curve AF, which indicates upstream values of F_d and h_m/d_0 where a stationary bore is possible in the flow approaching the obstacle. For these upstream conditions the steady flow near the obstacle can either be entirely supercritical, or have the stationary bore upstream of the obstacle leading to hydraulically controlled flow over the obstacle. The curve is obtained by setting $c_1=0$ in (1.7.1)-(1.7.5), resulting in

$$\frac{h_m}{d_0} = \frac{(8F_0^2+1)^{3/2}+1}{16F_0^2} - \frac{1}{4} - \frac{3}{2}F_0^{2/3}$$

If one performs the original version of Long's experiment in EAF, no upstream bore is found and the final steady state is the entirely supercritical flow, as in the upper left inset of Figure 1.7.2. The other alternative can be realized by starting with an obstacle of height $h_m>h_c$ (to the right of curve AE) and waiting until a hydraulically controlled flow is established. If the obstacle height is then gradually reduced to a value in the region EAF, the hydraulically controlled solution will persist.

A numerical demonstration of the implied hysteresis is shown in Figure 1.7.3. In frame (a) the obstacle of height $h_m>h_c$ is introduced, exciting an upstream bore. In (b) the obstacle has been lowered to a height $h_m<h_c$ such that h_m/d_o lies in region EAF. Here the bore continues to propagate upstream and the flow over the sill remains critical. Next the obstacle is lowered to point to the *left* of curve AF, causing the bore to reverse directions

and move downstream towards the obstacle (c). Eventually the bore moves past the obstacle (d) and a supercritical state is achieved.

Finally, the curve AD separates flows with and without hydraulic jumps attached to the downstream slope of the obstacle. For initial conditions lying below AD the jump would be positioned on the downstream slope of the obstacle. Above AD the jump would move downstream leaving supercritical flow behind. On AD the hydraulic jump will become stationary at the foot of the obstacle, as shown in Figure 1.7.1b. In order to find the obstacle height at which this last situation occurs one must piece together the segments of steady flow shown at sections 'a', 'b', 'c' and 'd' in the figure. There are 10 unknowns, including the depths and velocities at these four sections, the upstream bore speed, and the obstacle height. Four constraints are provided by the shock joining conditions across the bore and hydraulic jump. Also volume transport and energy (Bernoulli function) are conserved between sections 'a' and 'c' and between 'c' and 'b', providing 4 additional constraints. The final two constraints are provided by the condition of critical flow at the sill and the conservation of $R_{-}=v_o-2(gd_o)^{1/2}$ across the rarefaction wave that moves downstream of the obstacle. The algebra involved in the determination of the obstacle height from these ten relations is formidable.

The same sequence of events are seen in a laboratory demonstration that directs a supercritical current up a sloping channel with an open end (Figure 1.7.4). For a small channel slope, the current of dyed water remains in the supercritical state sketched in Slowly tilting the channel to progressively greater slopes is equivalent to gradually increasing h_m/d_0 with F_0 constant. If the slope is increased to the point where $h_m>h_c$, a bore forms at the edge of the open end (Panel *b*). The bore propagates to the left, down the slope, and establishes subcritical flow in the channel with critically controlled flow at the exit (Panel c). If the slope is then gradually decreased, this subcritical state persists. Eventually the slope is reduced to the point where bore forms at the source (Panel d). The bore and moves to the right, up the slope, a re-establishes supercritical flow in the channel (Panel e). This experiment is easy to set up in the classroom. All that is required is a small, handheld channel, and a system for circulating the water. The demonstration will show that the stationary upstream jump predicted along the curve AD of Figure 1.7.2 is unstable but can be manually balanced in the sloping channel with a little practice.

Exercises

1. Obtain equation 1.7.7 for the blocking height of the obstacle.

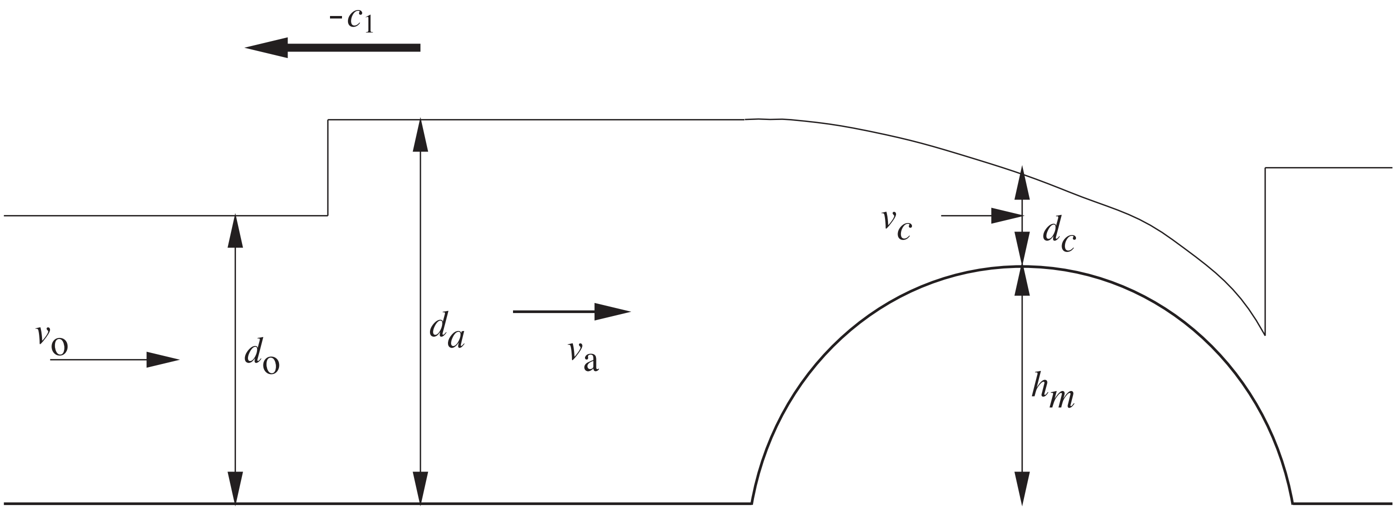
Figure Captions

1.7.1 The various transients generated by the introduction of an obstacle into a uniform stream when h_o exceeds the critical value h_c for upstream influence.

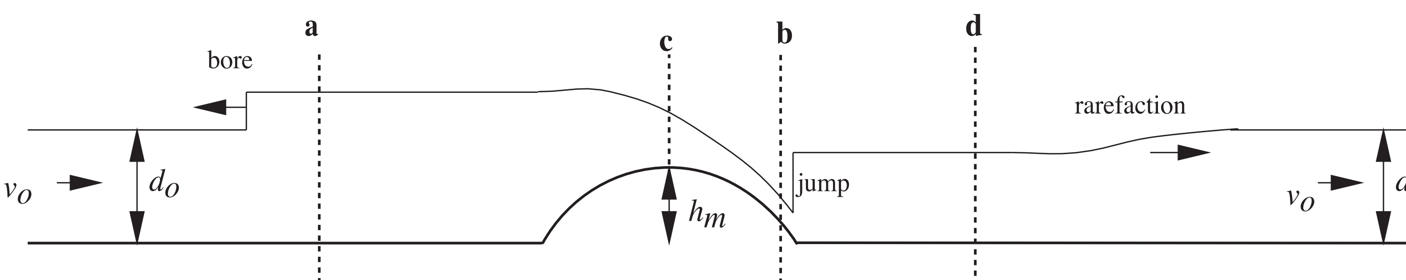
1.7.2 The various asymptotic regimes of the Long-type initial-value experiment in terms of the initial conditions.

1.7.3 Frame *a* shows the evolution of a shallow stream when an obstacle of height h_m is introduced into a moving stream of depth d_o, such that the initial conditions lie to the right of curve AE in Figure 1.7.2. The obstacle height is then lowered (Frame *b*) so that h_m/d_o lies in region EAF. Later h_m/d_o is decreased so as to lie to the left of curve AF (*c* and *d*). (From Pratt, 1983.)

1.7.4. Photograph of dyed water flowing up a sloping channel and spilling out at the right-hand end. The water is fed by a sluice gate with $F_0 = 5.6$ from a reservoir on the left. (a) Supercritical flow with $h_m/d_0 = 7.9$. (b) A hydraulic jump moving upstream trailed by subcritical flow $h_m/d_0 = 8.0$. (c) Subcritical flow in the entire channel $h_m/d_0 = 6.0$. (d) A hydraulic jump moving downstream trailed by supercritical flow $h_m/d_0 = 4.4$. (e) Supercritical flow in the entire channel a few seconds later. (From Baines and Whitehead, 2003.)



(a)



(b)

Figure 1.7.1

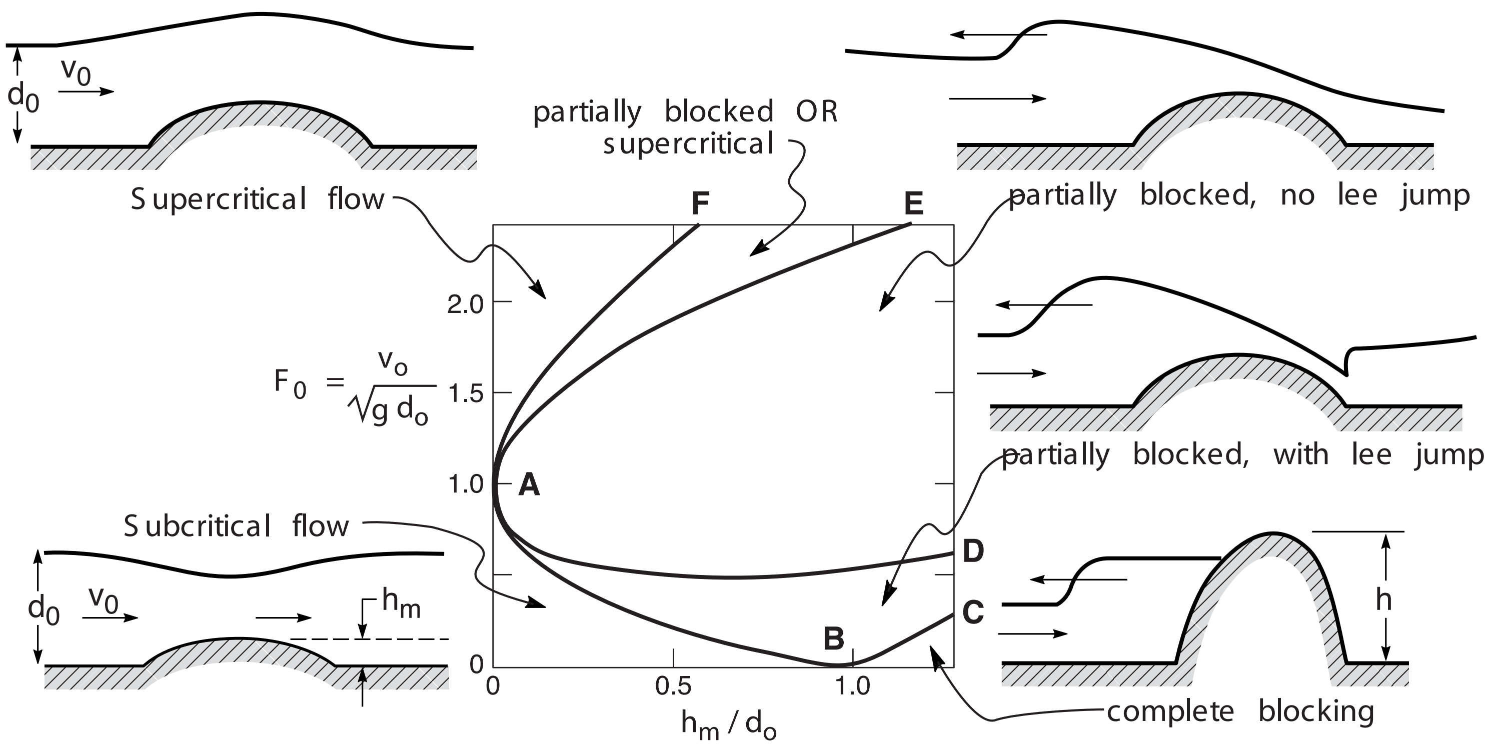

partially blocked OR supercritical
Supercritical flow
F
E
partially blocked, no lee jump
2.0
$F_0 = \frac{v_o}{\sqrt{g\,d_o}}$
1.5
1.0
A
partially blocked, with lee jump
Subcritical flow
D
C
B
0
0
0.5
1.0
h_m / d_o
complete blocking
d_0
v_0
h_m
h

Figure 1.7.2

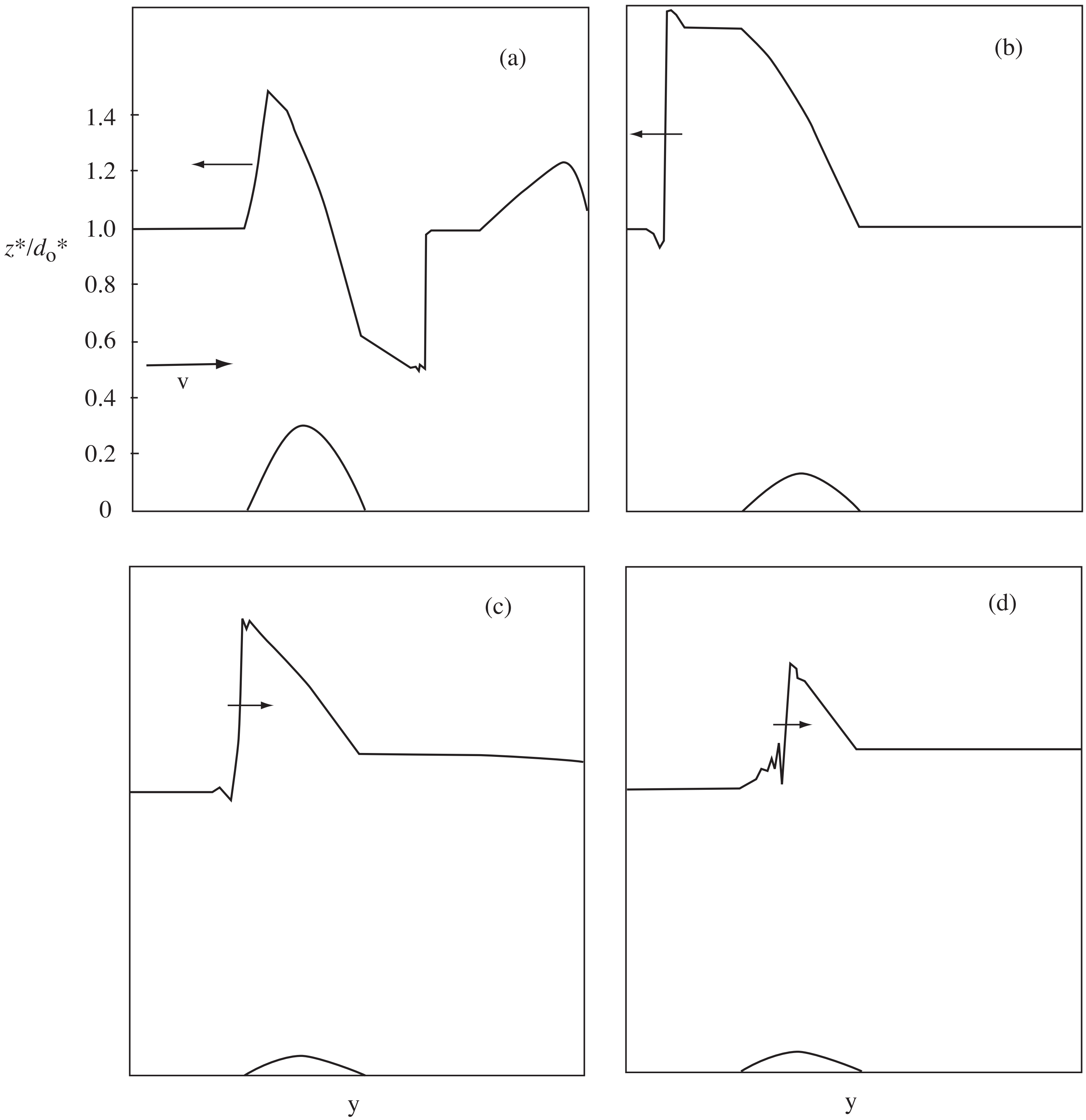
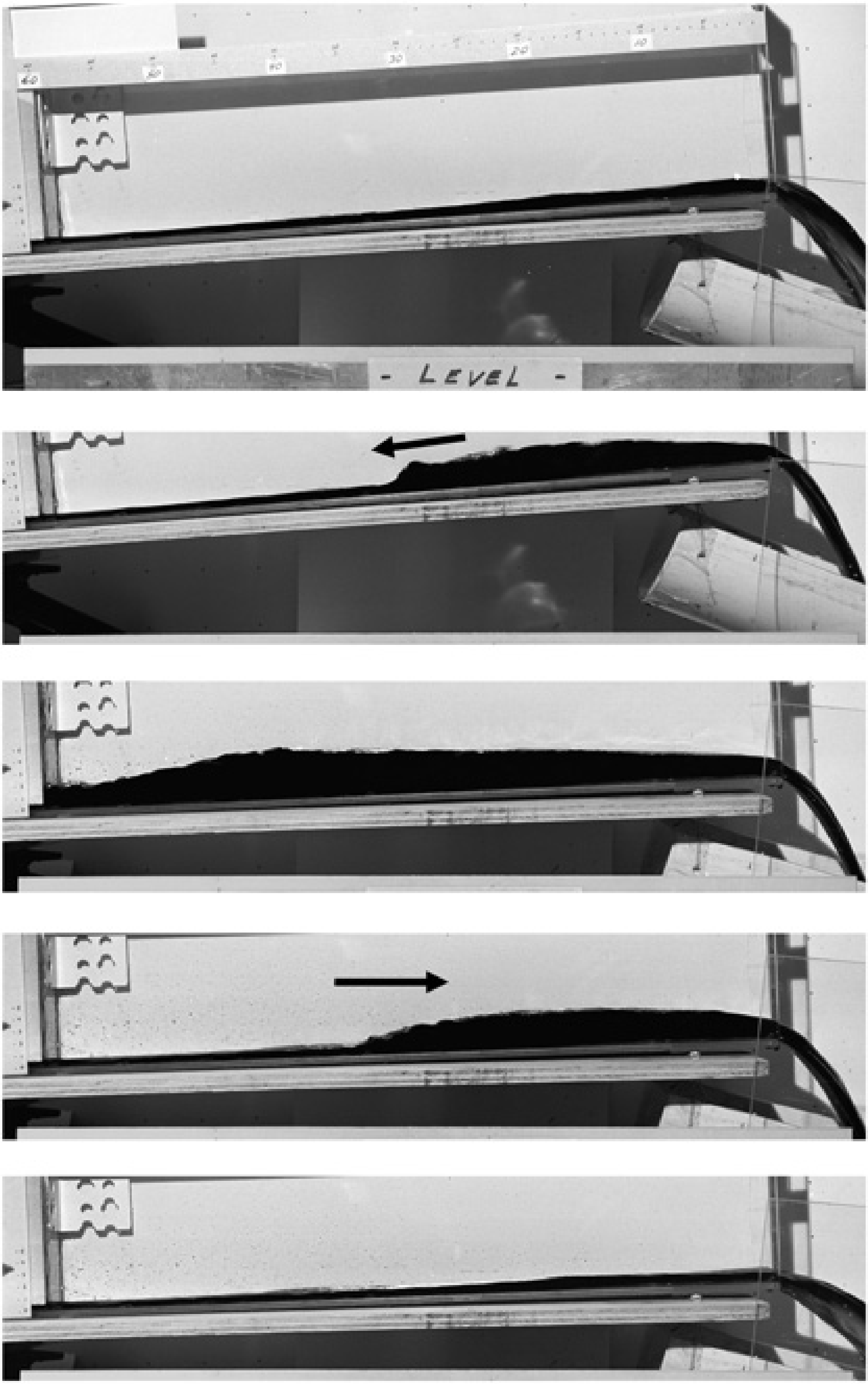


Figure 1.7.3



(a)

(b)

(c)

(d)

(e)