2.1 Semigeostrophic Flow in Rotating Channels.

The original models of rotating, hydraulically-driven currents were motivated by observations of deep overflows. The spillage of dense fluid over the sills of the Denmark Strait, the Faroe Bank Channel and other deep passages is suggestive of hydraulic control and one hope was that formulae used to estimate the volume outflow from a reservoir might be extended to these settings. To this end the whole volume of dense, overflowing fluid is treated as a single homogeneous layer with reduced gravity. In the Denmark Strait overflow example (Figures I7 and I8) this layer typically includes all fluid denser than σ_θ=27.9. The hypothetical homogeneous layer experiences strong, cross-channel variations in thickness and velocity, complications that can arise in engineering applications but are unavoidable where the earth's rotation is important. Much of the development of the theory of rotating hydraulics consists of attempts to come to grips with this extra degree of freedom. We shall trace this development beginning with early models of rotating-channel flow and show that hydraulic control and many of the other features reviewed in the first chapter remain present in one form or another. A number of novel features will also arise, including boundary layers, flow reversals, side-wall separation. In this presentation, we will use northern hemisphere flows as paradigms.

Another distinctive aspect of rotating hydraulics concerns the permissible waves. Under the usual assumption of gradual variations of the flow along its predominant direction, three types of waves arise. The first is the Kelvin wave, an edge wave closely related to the long gravity waves of the last chapter. The second is the frontal wave, which replaces the Kelvin wave when the edge of the flow is free to meander independently of sidewall boundaries. Frontal waves are sometimes referred to as Kelvin waves in the literature. The third is the potential vorticity wave, a disturbance that exists when gradients of potential vorticity, defined in this chapter, exist within the fluid. Nearly all analytical models of deep overflows assume that the potential vorticity is uniform within the flow, thereby eliminating this wave. We will consider only one model that does not. Free jets in the ocean and atmosphere are more dependent on potential vorticity dynamics and will be covered in Chapter 6.

In contrast to Chapter 1, where nearly all variables were dimensional, the present Chapter (and the remainder of the book) will depend primarily on dimensionless variables, and will frequently cite the dimensional representation of particular results. It becomes necessary to distinguish between the two formats, and we do so by assigning stars to dimensional quantities. There are some exceptions to this convention. Stars are not used, for example, for certain universally recognized dimensional quantities such as the Coriolis parameter, *f*, or for generic scales such *D* (for depth) and *L* (for length).

2.1 The semigeostrophic equations for homogeneous, rotating channel flow.

We consider homogeneous flows confined to a channel rotating with constant angular speed *f/2* in the horizontal plane. The coordinates (x^*,y^*) denote cross-channel and along-channel directions, (u^*,v^*) the corresponding velocity components, and (d^*,h^*) the fluid depth and bottom elevation. Provided the scale of x^*- and y^*-variations of d^* are large compared to the typical depth, the shallow water equations continue to apply. The dimensional version of these equations is

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}-fv^*=-g\frac{\partial d^*}{\partial x^*}-g\frac{\partial h^*}{\partial x^*}+F^{(x)*} \tag{2.1.1}$$

$$\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+fu^*=-g\frac{\partial d^*}{\partial y^*}-g\frac{\partial h^*}{\partial y^*}+F^{(y)*} \tag{2.1.2}$$

$$\frac{\partial d^*}{\partial t^*}+\frac{\partial(u^*d^*)}{\partial x^*}+\frac{\partial(v^*d^*)}{\partial y^*}=0\,. \tag{2.1.3}$$

Unspecified forcing and dissipation is contained in $\mathbf{F}=(F^{(x)*},F^{(y)*})$. For positive *f*, the channel rotation is counterclockwise looking down from above, as in the northern hemisphere. These equations apply to a homogeneous layer with a free surface or to the active lower layer of a '$1\frac{1}{2}$-layer' or 'equivalent barotropic' model. In the latter, *g* is reduced in proportion to the fractional density difference between the two layers. In such cases the upper boundary of the active layer will be referred to as 'the interface'.

For large-scale oceanic and atmospheric flows away from the equator and away from fronts and boundary layers, the forcing and dissipation terms and the terms expressing acceleration relative to the rotating earth are generally small in comparison to the Coriolis acceleration. The horizontal velocity for these types of flows is approximately geostrophic, or

$$fv^*\cong g\frac{\partial(d^*+h^*)}{\partial x^*}\qquad\text{and}\qquad fu^*\cong -g\frac{\partial(d^*+h^*)}{\partial y^*}$$

in the context of our shallow water model. These relations suggest that geostrophic flow moves parallel to lines of constant pressure, with high pressure to the right in the northern hemisphere. This situation was quite different for the flows treated in Chapter 1, in which the velocity is aligned with the pressure gradient and flow is accelerated from high to low pressure. For the deep overflows and strong atmospheric down-slope winds the acceleration of the flow down the pressure gradient is also a characteristic feature, suggesting a departure from the geostrophic balance.

To explore this issue further it is helpful to nondimensionalize variables. Define *D* as a scale characterizing the typical depth and *L* as a measure of the horizontal distance over which along-channel variations take place. Also take $(gD)^{1/2}$ as a scale for v^*, anticipating that the gravity wave speed will continue to be a factor in the dynamics of hydraulically controlled states and that such states will require velocities as large as this speed. A natural scale for t^* is therefore $L/(gD)^{1/2}$. As a width scale, we pick $(gD)^{1/2}/f$, which is the Rossby radius of deformation based on the depth scale D. For readers not familiar with the theory of rotating fluids, $2\pi(gD)^{1/2}/f$ is the distance a long gravity wave [with speed $(gD)^{1/2}$] will travel in an inertial period $2\pi/f$. It is the distance the wave must travel before it is influenced by the earth's rotation. Motions with much smaller length or time scales are generally not influenced by rotation. The Rossby radius appears as a natural width scale for boundary currents and boundary-trapped waves. With these choices, the cross-channel velocity scale $(gD)/fL$ is suggested by balancing the second and third terms in (2.1.3). The dimensionless variables are therefore

$$x=\frac{x^*f}{(gD)^{1/2}},\quad y=\frac{y^*}{L},\quad t=\frac{t^*(gD)^{1/2}}{L}$$

$$v=\frac{v^*}{(gD)^{1/2}},\quad u=\frac{fLu^*}{gD},\quad d=\frac{d^*}{D},\quad h=\frac{h^*}{D},\quad \mathbf{F}=\frac{L\mathbf{F}^*}{gD}. \tag{2.1.4}$$

Substitution into (2.1.1-3) leads to

$$\delta^2\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-v=-\frac{\partial d}{\partial x}-\frac{\partial h}{\partial x}+\delta F^{(x)} \tag{2.1.5}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+u=-\frac{\partial d}{\partial y}-\frac{\partial h}{\partial y}+F^{(y)} \tag{2.1.6}$$

$$\frac{\partial d}{\partial t}+\frac{\partial (ud)}{\partial x}+\frac{\partial (vd)}{\partial y}=0 \tag{2.1.7}$$

where $\delta=(gD)^{1/2}/fL$ is the ratio of the width scale of the flow to L: a horizontal aspect ratio.

The limit $\delta\rightarrow 0$ leads to a geostrophic balance in the cross-channel (x-) direction but not the along channel direction. The along-channel velocity v is geostrophically balanced but the cross-channel velocity u is not. The flow in this limit is therefore referred to as *semigeostrophic*. The semigeostrophic approximation requires that variations of the flow along the channel are gradual in comparison with variations across the channel. In particular, the interface may slope steeply across the channel but can do so only mildly along the channel. The along channel velocity component v is therefore directed nearly perpendicular to the pressure gradient. As (2.1.6) suggests, the (weaker) along-channel pressure gradient *does* lead to acceleration in the same direction, but this occurs over a distance L large compared to the cross-stream scale δL.

Semigeostrophic and *quasigeostrophic* models should not be confused. In the latter, both of the horizontal velocity components are geostrophically balanced, at least to a first approximation, and variations in the depth or layer thickness are required to be slight. Time variations occur on a scale much longer than $1/f$. Quasigeostophic models form the basis for much of the theory of broad scale waves and circulation in the ocean and atmosphere (e.g. Pedlosky, 1987). Hydraulic effects with respect gravity waves cannot occur because these waves are filtered by the quasigeostrophic approximation.

Vorticity and potential vorticity are conceptually and computationally central to rotating flows. For shallow homogeneous flow, the discussion is simplified by the fact that the horizontal velocity is z-independent, so that the fluid moves in vertical columns. Vorticity and potential vorticity are therefore assigned to fluid columns as a whole. If the curl of the shallow water momentum equations (i.e. $\partial(2.1.2)/\partial x^* - \partial(2.1.1)/\partial y^*$) is taken and (2.1.3) is used to eliminate the divergence of the horizontal velocity from the resulting expression, the following conservation law for potential vorticity can be obtained:

$$\frac{d^* q^*}{dt^*} = \frac{\mathbf{k}\cdot\nabla^*\times\mathbf{F}^*}{d^*}. \qquad (2.1.8)$$

Here $\frac{d^*}{dt^*} = \frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*}$, $\mathbf{k}$ is the vertical unit vector, and

$$q^* = \frac{f+\zeta^*}{d^*} \quad . \qquad (2.1.9)$$

The *relative vorticity* $\zeta^* = \frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*}$ is the vorticity of a fluid column as seen in the rotating frame of reference. The *absolute vorticity* is the total vorticity ζ^*+f of the column. The *potential vorticity* q^* is simply the absolute vorticity divided by the column thickness d^*. If the forcing and dissipation have no curl ($\nabla^*\times\mathbf{F}^* = 0$) the potential vorticity of the material column remains constant. Conservation of potential vorticity is a consequence of angular momentum conservation; if the column thickness d^* increases, conservation of mass requires the cross-sectional area of the column to decrease, and the column must spin more rapidly to compensate for a decreased moment of inertia.

It is sometimes convenient to represent the potential vorticity as

$$q^* = \frac{f+\zeta^*}{d^*} = \frac{f}{D_\infty}$$

where D_∞ is known as the *potential depth*. In the absence of forcing or dissipation, each fluid column owns its own time-independent potential depth. To interpret this quantity, consider a column with relative vorticity ζ^* (also $=q^*d^*-f$ by the definition of q^*). Next alter the column thickness d^* to the value f/q^*, so that ζ^* vanishes. This new thickness *is* the potential thickness D_∞. This interpretation is limited by the fact that D_∞ may be negative, making it physically impossible to remove ζ^* by stretching. Most of the applications we will deal with have positive potential depth.

The nondimensional versions of (2.1.8) and (2.1.9) are

$$\frac{dq}{dt} = \frac{\dfrac{\partial F^{(y)}}{\partial x} - \delta \dfrac{\partial F^{(x)}}{\partial y}}{d} \tag{2.1.10}$$

and

$$q = \frac{1 + \dfrac{\partial v}{\partial x} - \delta^2 \dfrac{\partial u}{\partial y}}{d}. \tag{2.1.11}$$

In the semigeostrophic limit $\delta \to 0$:

$$v = \frac{\partial d}{\partial x} + \frac{\partial h}{\partial x}. \tag{2.1.12}$$

and

$$q = \frac{1 + \dfrac{\partial v}{\partial x}}{d}. \qquad (2.1.13).$$

The last two relations can be combined, yielding an equation for the x-variation in depth

$$\frac{\partial^2 d}{\partial x^2} - qd = -1 - \frac{\partial^2 h}{\partial x^2}. \tag{2.1.14}$$

If q=constant the above equation can easily be solved, reducing the calculation to a two-dimensional problem (in y and t). This situation arises if q is initially uniform throughout the fluid and no forcing or dissipation is present.

Two other forms of the shallow water momentum equations that will prove very helpful. One is

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (f + \zeta^*)\mathbf{k} \times \mathbf{u}^* = -\nabla B^* + \mathbf{F}^*, \tag{2.1.15}$$

where

$$B^* = \frac{u^{*2} + v^{*2}}{2} + g(d^* + h^*) \qquad (2.1.16)$$

is the two-dimensional Bernoulli function. The dimensionless form of the latter is $B = B^*/gD = \frac{1}{2}(\delta^2 u^2 + v^2) + d + h$. In the semigeostrophic limit B formally reduces to its one-dimensional equivalent $v^2/2 + g(d+h)$. The second version of interest is the depth-integrated or 'flux' form, obtained by multiplication of (2.1.1) and (2.1.2) by d^*, rearrangement of some derivatives, and use (2.1.3). The results:

$$\frac{\partial(d^* u^*)}{\partial t^*} + \frac{\partial}{\partial x^*}\left(d^* u^{*2} + \tfrac{1}{2} g d^{*2}\right) + \frac{\partial}{\partial y^*}(u^* v^* d^*) - f v^* d^* = -g d^* \frac{\partial h^*}{\partial x^*} + d^* F^{(x)*} \qquad (2.1.17a)$$

and

$$\frac{\partial(d^* v^*)}{\partial t^*} + \frac{\partial}{\partial y^*}\left(d^* v^{*2} + \tfrac{1}{2} g d^{*2}\right) + \frac{\partial}{\partial y^*}(u^* v^* d^*) + f d^* u^* = -g d^* \frac{\partial h^*}{\partial y^*} + d^* F^{(y)*} \qquad (2.1.17b)$$

are used in the analysis of hydraulic jumps, form drag and other applications where the total momentum over the water column is at issue.

If the flow is steady $(\partial/\partial t^* = 0)$, the continuity equation (2.1.3) implies the existence of a *transport stream function* $\psi^*(x,y)$ such that

$$v^* d^* = \frac{\partial \psi^*}{\partial x^*} \text{ and } -u^* d^* = \frac{\partial \psi^*}{\partial y^*}$$

The total volume transport Q^* is the value of ψ^* on the right-hand edge of the flow (facing positive y^*) minus ψ^* on the left wall. If, in addition, there is no forcing or dissipation (**F***=0) then (2.1.15) can be written

$$\frac{(f + \zeta^*)}{d^*}\mathbf{k} \times \mathbf{u}^* d^* = -\nabla B^* \qquad (2.1.18)$$

or $q^* \nabla \psi^* = \nabla B^*$. Thus the Bernoulli function is conserved along streamlines:

$$B^* = B^*(\psi^*)$$

and

$$q^* = \frac{dB^*}{d\psi^*}\,. \qquad (2.1.19)$$

This remarkable link between energy and potential vorticity is one of the central constraints used in hydraulic theories for two-dimensional flow. As shown by Crocco (1937), the relationship (2.1.19) holds in more general settings.

In the steady sill flows discussed in Chapter 1, the reservoir state is specified by the values of Q^* and B^*, the fundamental conserved quantities of the one-dimensional flow. Discussion of the present generalization often centers on three conserved quantities: the *functions* $B^*(\psi^*)$, $q^*(\psi^*)$ and the constant Q^*. Crocco's theorem shows that these three are not independent; specification of $B^*(\psi^*)$ and of the range of ψ^* allows $q^*(\psi^*)$ to be completely determined.

We have already touched on the different types of long (semigeostrophic) waves that arise in rotating channel flows. Kelvin waves and their frontal relatives depend on the combined effects of rotation and gravity and are important to the hydraulics of gravity-driven flows. Potential vorticity waves can exist in flows with neither gravity nor background rotation. Their dynamics involve vortex induction mechanics that can arise when the potential vorticity of the fluid flow varies spatially. If the long-wave assumption is relaxed, inertia-gravity (Poincaré) waves come into play. They are not important in traditional models of rotating hydraulics, but they are important for a range of transient phenomena generally considered to be part of hydraulics. We now discuss some of the linear properties of these waves where they arise as small perturbations from a resting state. Nonlinear steepening and other finite amplitude effects will be treated in later sections.

Consider the shallow water equations, linearized about a state of rest with d=1 and $\boldsymbol{F}$=h=0. Take $d = 1 + \eta$, with $|\eta|$<<1; assume u<<1 and v<<1; and neglect terms quadratic in η, v etc. in (2.1.5-2.1.7) to obtain

$$\delta^2 \frac{\partial u}{\partial t} - v = -\frac{\partial \eta}{\partial x} \qquad (2.1.20)$$

$$\frac{\partial v}{\partial t} + u = -\frac{\partial \eta}{\partial y} \qquad (2.1.21)$$

and

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \qquad (2.1.22)$$

The corresponding potential vorticity equation, which can be obtained directly from the above or simply by linearizing the nondissipative version of (2.1.8), is

$$\frac{\partial}{\partial t}\left[\frac{\partial v}{\partial x}-\delta^2\frac{\partial u}{\partial y}-\eta\right]=0$$

or

$$\frac{\partial v}{\partial x}-\delta^2\frac{\partial u}{\partial y}-\eta=\frac{\partial v_o}{\partial x}-\delta^2\frac{\partial u_o}{\partial y}-\eta_o, \qquad (2.1.23)$$

where $(\ )_o$ indicates an initial value. The last equation indicates that the linearized potential vorticity, equal to the relative vorticity $\partial v/\partial x-\delta^2\partial u/\partial y$ plus the stretching contribution $-\eta$, is conserved at each (x,y).

The left hand side of (2.1.23) can be expressed in any of the three variables u,v, or η by using (2.1.20-2.1.22) to eliminate the remaining two. For example the equation for η is

$$\frac{\partial^2\eta}{\partial x^2}+\delta^2\frac{\partial^2\eta}{\partial y^2}-\delta^2\frac{\partial^2\eta}{\partial t^2}-\eta=\frac{\partial v_o}{\partial x}-\delta^2\frac{\partial u_o}{\partial y}-\eta_o. \qquad (2.1.24)$$

For an arbitrary initial disturbance the resulting flow will consist of two parts. The first is a steady flow whose potential vorticity is given by the potential vorticity of the initial disturbance. This flow is obtained by finding a steady solution to (2.1.24). The second component consists of waves that are generated as a result of the unbalanced part of the initial flow. Individually, these waves are solutions to the homogeneous version of (2.1.24) subject to the boundary condition

$$\frac{\partial^2\eta}{\partial x\partial t}=-\frac{\partial\eta}{\partial y}, \qquad (x=\pm w/2) \qquad (2.1.25)$$

obtained by evaluating (2.1.20) and (2.1.21) at the sidewalls, where u=0, and eliminating v from the result.

Assuming traveling waves of the form $\eta=\mathrm{Re}\left[aN(x)e^{i(ly-\omega t)}\right]$, where ω is the frequency and l is the longitudinal wave number, one finds two distinct solutions (Gill, 1982 or Pedlosky 2003), both of which were discovered by Kelvin (1879). The first, named after Poincaré (1910), has an oscillatory structure in x:

$$N_n(x)=\cos(k_nx)+b_n\sin(k_nx) \qquad (2.1.26)$$

where $k_n=n\pi/w$, and $b_n=-\omega_nk_n/l$ (n=odd) or $b_n=l/\omega_nk_n$ (n=even). The frequency satisfies the dispersion relation

$$\delta^2\omega^2=\frac{n^2\pi^2}{w^2}+\delta^2l^2+1\,(n=1,2,3,...),$$

the dimesional form of which is

$$\omega^{*2} = gD(\frac{n^2\pi^2}{w^{*2}} + l^{*2}) + f^2 \qquad (2.1.27)$$

where D is the background depth.

Poincaré waves can be better understood by first considering a long gravity wave propagating in an arbitrary direction on an infinite, nonrotating plane. The form of the wave is given by $\eta^* = \mathrm{Re}\left[a^* e^{i(k^*x^*+l^*y-\omega^*t^*)}\right]$, where k^* and l^* represent the wave numbers. The dispersion relation for this wave is given in dimensional terms by (2.1.27) with f=0 and with k^* replaced by the discrete wave number $(n^2\pi^2/w^*)^2$. Next consider a second wave with wave numbers $(-k^*, l^*)$ and therefore having the same frequency as the first wave. If the second wave has the same amplitude a as the first, a superposition of the two leads to a u^* field proportional to $\mathrm{Re}\left[a(e^{i(k^*x^*+l^*y^*-\omega^*t^*)} - e^{i(-k^*x^*+l^*y^*-\omega^*t^*)})\right] = \mathrm{Re}\left[2aie^{i(l^*y^*-\omega^*t^*)}\sin k^*x^*\right]$. Since u^* is zero whenever k^*x^* is an integer multiple of π, the waves satisfy the side-wall boundary conditions in a channel with side walls at $x^* = \pm w^*/2$ provided that k^* is chosen to be $2n\pi/w^*$. These waves are sometimes called oblique gravity waves and their cross-channel structure is said to be *standing*. Poincaré waves are rotationally modified versions of these waves.

The second class consists of edge waves named after Kelvin himself. The cross-channel structure and dispersion relation are given by

$$N_\pm(x) = \frac{\sinh(x) \pm \cosh(x)}{\sinh(\frac{1}{2}w)} \qquad (2.1.28)$$

and

$$\omega_\pm = \pm l \text{ or}$$

$$\omega^*_\pm = \pm(gD)^{1/2} l^* \qquad (2.1.29)$$

Kelvin waves have a boundary layer structure that becomes apparent when the channel width is much wider than the deformation radius. Taking the limit w>>1 (equivalently $w^* >> (gD)^{1/2}/f$) in (2.1.28) leads to

$$N_+^*(x^*) \propto N^*(\tfrac{1}{2}w^*)e^{(x^*-\frac{1}{2}w^*)f/(gD)^{1/2}}$$

and

$$N_-^*(x^*) \propto N^*(-\tfrac{1}{2}w^*)e^{-(x^*+\frac{1}{2}w^*)f/(gD)^{1/2}} .$$

The first solution corresponds to a wave propagating in the positive y-direction at speed $(gD)^{1/2}$ and trapped to the wall at $x^*=w^*/2$. The trapping distance is the Rossby radius of deformation based on the background depth D. The other wave moves in the opposite direction and is trapped to the wall at $x^*=-w^*/2$. In the limit of weak rotation, $N_\pm^*$ becomes constant and the Kelvin waves reduce to x-independent, long gravity waves propagating along the channel. A further distinguishing property of linear Kelvin waves is that the cross-channel velocity u is identically zero.

Kelvin waves are nondispersive, meaning that the phase speed c^* does not depend on the wave number l^*. The wave frequency $\omega^*=c^*l^*$ is proportional to l^* and therefore the group velocity $\partial\omega^*/\partial l^*$ is equal to c^*. In Chapter 1, we described the topographic resonance that can occur when a background flow is critical $c^*=0$ with respect to a nondispersive wave. A bottom slope or other stationary forcing introduces disturbance energy that cannot propagate away. The disturbance amplitude grows and becomes large and sufficiently nonlinear to break away, leading to fundamental changes in the upstream flow. We expect that Kelvin waves will play an important role in the upstream influence of rotating channels flows.

Poincaré waves are not admitted under semigeostrophic dynamics, a result that can be shown by taking ($\delta\to 0$) in (2.1.27). The limiting condition $(\frac{n^2\pi^2}{w^2}+1=0)$ cannot be satisfied for real n. Since most simple models of the hydraulics of rotating flow in a channel or along a coast use the semigeostrophic approximation, Poincaré waves do not arise. However, there are a few models of unbounded flows for which hydraulic effects arise (e.g. Section 3.8). These effects involve Poincaré waves with short wave lengths ($l\to\infty$), for which (2.1.27) reduces to $c=\omega/l=\pm 1$ (or $\omega^*=\pm(gD)^{1/2}$). In this limit the waves behave like nonrotating gravity waves and can be considered nondispersive if propagation is somehow limited to a single direction.

The restoring mechanism for Poincaré and Kelvin relies on gravity and a free surface or interface. Potential vorticity waves, on the other hand, rely on gradients of potential vorticity within the fluid. One can describe this effect by modifying the above example to include a lateral bottom slope $\partial h^*/\partial x^*=-S=const$. For simplicity, we will eliminate the gravitational restoring mechanism by placing a rigid lid on the top of the fluid. The resting basic state now contains a potential vorticity gradient associated with the variable depth alone. If D is the layer thickness at mid-channel ($x^*=0$) and if the bottom and surface tilt lead to only slight variations of h^* about D, then the potential vorticity of the ambient fluid is

$$q^*=\frac{f+\partial v^*/\partial x^*}{d^*}=\frac{f}{D+sx^*}\cong\frac{f}{D}-\left(\frac{sf}{D^2}\right)x^*.$$

Under these conditions the flow will support potential vorticity waves with phase speeds given by

$$c^* = -\left(\frac{Sf}{D}\right)\frac{1}{(n^2\pi^2 / w^{*2}) + l^{*2}}, \quad n=1,2,3,\cdots$$

In the long wave limit ($w^*l^*\to 0$) the waves are nondispersive:

$$c^* = -\frac{Sfw^{*2}}{n^2\pi^2 D} = \left(\frac{dq^*}{dx^*}\right)\frac{w^{*2}\,D}{n^2\pi^2}, \quad n=1,2,3,\cdots \qquad (2.1.30)$$

where w^* is the channel width and $\frac{dq^*}{dx^*} = -\frac{Sf}{D^2}$. This example is discussed fully by Pedlosky (2003). For positive S, $\partial q^*/\partial x^* < 0$ and higher potential vorticity is found on the left-hand side (facing positive y^*) of the channel. In this case the propagation tendency of the waves is towards negative y^*.

The waves produced in the last example are called topographic Rossby waves since the background potential vorticity gradient was created by a sloping bottom. More generally, steady flows with nontrivial depth and vorticity distributions have potential vorticity gradients and will support potential vorticity waves, although some of these may be unstable. The nondispersive character of the long waves is indicative of their ability to transmit upstream influence, an effect that will be demonstrated in later sections.

Exercises

1) *Dissipation and vorticity flux.*

(a) By taking the curl of the shallow water momentum equations (2.1.15) obtain the vorticity equation

$$\frac{\partial \zeta_a^*}{\partial t^*} + \nabla\cdot(\mathbf{u}^*\zeta_a^*) = \mathbf{k}\cdot(\nabla\times\mathbf{F}^*), \qquad (2.1.31)$$

where $\zeta_a^* = f + \zeta^*$ is the total (or *absolute)* vorticity of a fluid column.

(b) Define $\mathbf{J}_n^*=\mathbf{k}\times\mathbf{F}^*$ and write $\mathbf{k}\cdot(\nabla\times\mathbf{F}^*) = -\nabla\cdot\mathbf{J}_n^*$, so that (2.1.31) becomes

$$\frac{\partial \zeta_a^*}{\partial t^*} + \nabla\cdot(\mathbf{u}^*\zeta_a^* + \mathbf{J}_n^*) = 0\,. \qquad (2.1.32)$$

The quantity $\mathbf{u}^*\zeta_a^* + \mathbf{J}_n^*$ may be interpreted as the total flux of absolute vorticity, the term $\mathbf{u}^*\zeta_a^*$ accounting for the advective part of the flux and the term $\mathbf{J}_n^*$ accounting for the dissipative flux.

(c) By taking the cross product of **k** with the steady version of (2.1.15) obtain the relation

$$\mathbf{k}\times\nabla B^* = \mathbf{u}^*\zeta_a^* + \mathbf{J}_n^*. \qquad (2.1.33)$$

By comparing this with the relation $\mathbf{k}\times\nabla\psi^* = \mathbf{u}^*$ interpret B^* as a streamfunction for the total vorticity flux. Further show that the derivative of B^* along streamlines gives a vorticity flux that is entirely due to dissipation, whereas the derivative of B^* in the direction normal to streamlines gives a flux that is partly due to dissipation and partly due to advection.

[The main ideas developed in this exercise are due to Schär and Smith (1993).]

2) Equation (2.1.24) paves the way for solution to the linear shallow water equations in terms of η. Show that the equivalent equations for u and v are given by

$$\frac{\partial^2 u}{\partial x^2} + \delta^2\frac{\partial^2 u}{\partial y^2} - \delta^2\frac{\partial^2 u}{\partial t^2} - u = -\frac{\partial}{\partial y}\left[\frac{\partial v_o}{\partial x} - \delta^2\frac{\partial u_o}{\partial y} - \eta_o\right] \qquad (2.1.34)$$

and

$$\frac{\partial^2 v}{\partial x^2} + \delta^2\frac{\partial^2 v}{\partial y^2} - \delta^2\frac{\partial^2 v}{\partial t^2} - v = \frac{\partial}{\partial x}\left[\frac{\partial v_o}{\partial x} - \delta^2\frac{\partial u_o}{\partial y} - \eta_o\right]. \qquad (2.1.35)$$