2.6 Uniform potential vorticity flow, revisited.

Some aspects of the Gill's formulation for uniform potential vorticity flow from a wide basin can be unintuitive. One hobgoblin is the scaling choice $D=(fQ^*/2g)^{1/2}$, which causes the volume flux to be hidden in the dimensionless forms of the governing equations. Another difficulty involves the choice of ψ_i as a parameter for the upstream flow. Although intuitively satisfying, this quantity (or its dimensional version) is difficult to measure in typical deep-sea settings. More recent investigators have explored other choices of upstream parameters and have shown that certain choices can lead to a simplified, and in some cases more data friendly, formulation.

If the depth scale D is left unspecified for the moment, then the continuity equation (2.5.1) reverts to its earlier form

$$2\hat{d}\bar{d}=Q \qquad (2.6.1)$$

while the Bernoulli relation (2.5.3) becomes

$$T^2(\bar{d}-q^{-1})^2+\frac{Q^2}{4T^2\bar{d}^2}+2q^{-1}(\bar{d}+h)=2q^{-1}\bar{B}\,. \qquad (2.6.2)$$

The left-hand side of (2.6.2) may be treated as a Gill-type functional and the critical condition may be found by setting its derivative with respect to $\bar{d}$ to zero. The result is a slightly modified form of (2.5.6):

$$(1-T_c^2)q^{-1}+T_c^2\bar{d}_c=Q^2\bar{d}_c^{-3}T_c^{-2}/4\,. \qquad (2.6.3)$$

The parameter $\bar{B}$, the average of the side-wall Bernoulli functions, was vanquished by Gill in preference to ψ_i. One alternative [Whitehead and Salzig, 2001] is to use the value $B=B_R$ along the right side wall. We will generalize their discussion and also consider the value B_L on the left wall. Integration of the relation $dB/d\psi=q$ across the channel then yields

$$\bar{B}=\tfrac{1}{2}[B_R+B_L]=B_R-\tfrac{1}{2}qQ=B_L+\tfrac{1}{2}qQ\,. \qquad (2.6.4)$$

If (2.6.2) is now evaluated at the sill ($h=h_c$) (2.6.4) can be used to write the result in the various forms

$$T_c^2(\bar{d}_c - q^{-1})^2 + \frac{Q^2}{4T_c^2\bar{d}_c^{\,2}} + 2q^{-1}\bar{d}_c = \begin{cases} 2q^{-1}(B_R - h_c) - Q \\ \text{or } 2q^{-1}(\bar{B} - h_c) \\ \text{or } 2q^{-1}(B_L - h_c) - Q \end{cases} \qquad (2.6.5a,b,c)$$

So far the depth scale D has remained arbitrary, but it is now possible to select it in a way that reduces the dependence of Q on the upstream state to a single parameter. For example, suppose that the first form of the above relation is chosen and that $B_R - h_c$ is set to unity, which is equivalent to

$$D = g^{-1}B*_R - h_c*. \qquad (2.6.6)$$

The only remaining flow parameter is q. Elimination of $\bar{d}_c$ between (2.6.3) and the newly scaled form of (2.6.5a) determines Q in terms of only w_c and q. In dimensional terms the volume flow rate is $Q* = gD^2f^{-1}Q(q,w_c) = (B_R* - gh_c*)^2(gf)^{-1}Q(q,w_c)$. A further conceptual simplification can be made by imagining that the flow stagnates along the right-hand (northern hemisphere) wall at some point upstream of the sill. The surface elevation above the sill at this location is $\Delta z_R* = g^{-1}B_R* - h_c*$, just the depth scale D. The transport relation may therefore be written in the form

$$Q* = \frac{g(\Delta z_R*)^2}{f}Q(q,w_c). \qquad (2.6.7a)$$

The function $Q(q,w_c)$, which is obtained by eliminating $\bar{d}_c$ between 2.5.6a (with $B_R - h_c = 1$) and (2.6.3), is contoured in Figure 2.6.1. One facet that stands out is the insensitivity of Q to the potential vorticity q when the sill width is moderate or small ($w_c<2$).

The discussion thus far has been restricted to attached flow. However, the form of Q for separated flow at the control section can be deduced from a simple argument. We first note that the dimensional geostrophic transport at a separated section, regardless of the potential vorticity distribution, is $Q* = g(d*(w*/2))^2/2f$, where $d_c*(w*/2)$ is the depth at the right wall. It has also been shown (Exercise 2 of Section 2.5) that the velocity at the right wall is zero when the flow is both critical and separated and the potential vorticity is uniform. The value of the right-wall Bernoulli function, generally $g(h_c* + \Delta z_r*)$, must then be $g(h_c* + d_c*(w*/2))$. The right wall depth is therefore Δz_R* and the transport is given by (2.6.7a) with Q=1/2, or

$$Q* = \frac{g(\Delta z_R*)^2}{2f}. \qquad (2.6.7b)$$

In fact, this formula is valid for a larger class of solutions than those with uniform q. It was argued in Section 2.3 that any separated flow with v=0 along the right wall is hydraulically critical. The arguments leading to (2.6.7b) remain valid for any such flow. We also leave it to the reader to prove (see Exercise 1) that the deceptively complex relation (2.5.14) obtained by Gill is equivalent to (2.6.7b). The reader may also wish to note the similarity of (2.6.7b) to the zero potential vorticity transport relation (2.4.15).

One could have made similar simplifications, using the second or third forms of (2.6.5) in connection with different choices in D. As shown by Iacono (2006), an advantage in choosing the third form, which utilizes the left-wall Bernoulli function, is that it leads to a closed form analytical expression for the transport (see Exercise 4). However, there are reasons to prefer the right-wall Bernoulli function $B_R{}^*$, or equivalently $\Delta z_R{}^*$, as the upstream parameter. One is that stagnant or sluggish upstreams regions are found in models and laboratory experiments along the right wall. This topic is discussed in detail in Sections 2.7 and 2.14. Also, since Kelvin wave propagation in the basin is positive (downstream) along the right wall, changes in the flow far upstream are communicated to the strait along this wall. Information propagation along the left wall must be upstream since the flow there is subcritical. Information is therefore routed counterclockwise around the edge of the basin, making it reasonable to believe that right wall information can be specified independently. Of course, these ideas require modification when the upstream basin is closed.

In the Gill model, specification of the upstream state requires two dimensionless parameters, q and Gill's ψ_i. With the present scaling and parameter choices, upstream information is formally specified by q alone. This treatment is more elegant, but it hides the fact that a particular Q corresponds to a whole range of upstream states, each with its own distribution of boundary layer fluxes. That is, each point in Figure 2.6.1 corresponds to a range of upstream flows with the same Q. To specify the full upstream state at such a point, one must know the second upstream parameter B_R, which is hidden in the scaling. It is advantageous to use the closely related parameter qB_R which, in dimensional terms, is the ratio of B_R* to the Bernoulli function gD_∞ in the basin interior. Large values of qB_R therefore have relatively energetic right-wall boundary currents, whereas qB_R=1 has no right-wall current at all. Consider the range of upstream states possible for the setting q=0.8 and w_c=1, indicated by a star in Figure 2.6.1. The case qB_R=7.0 (Figure 2.6.2a) confirms the expected energetic nature of the right-wall flow. In fact the boundary layer transport is more than can squeeze through the critical section; most of it returns along the left wall. The high kinetic energy along the right wall allows the flow to rise up and pass over a sill whose elevation is much greater than the interior surface elevation. As qB_R decreases, the right-wall boundary layer weakens (frame b) and disappears (frame c for qB_R=1). At the lowest value of qB_R (frame d), the right-wall flow reverses and the left layer carries all of the positive flux. When qB_R reaches its minimum possible value $q^2Q+\frac{1}{2}$ (see Exercise 2 and the dashed contours in Figure 2.6.1), the surface at the left wall grounds and the flow separates.

To apply Figure (2.6.1) in laboratory or field situations, it is helpful to write out the dimensional form of (2.6.7)

$$Q^* = \frac{g(\Delta z_R^*)^2}{f} Q\left(\frac{\Delta z_R^*}{D_\infty}, \frac{f w_c^*}{(g\Delta z_R^*)^{1/2}}\right) \quad (\Delta z_R^* = g^{-1} B_R^* - h_c^*) \qquad (2.6.8)$$

and thereby acknowledge that *two* dimensional upstream parameters (B_R^* and D_∞) are required, along with the sill height and width, to determine the dimensional transport. (Gill requires ψ_i^* and D_∞ as upstream parameters.) Determination of the value of B_R^* generally means that a velocity measurement must be made. However this problem is alleviated if a region of quiescent flow along the right wall can be found: there the surface elevation above the sill is exactly Δz_R^*. Alternatively, B_R^* can be related to other properties that may be more easily measured. Some of these can be derived from the expressions for the boundary layer flow in the vicinity of the right wall:

$$d^*(x,y) = D_\infty + (d^*(w/2,y) - D_\infty) e^{(x^* - w^*/2) f/(gD_\infty)^{1/2}} \qquad (2.6.9)$$

and

$$v^*(x^*,y^*) = \left(\frac{g}{D_\infty}\right)^{1/2} (d^*(w/2,y) - D_\infty) e^{(x^* - w^*/2) f/(gD_\infty)^{1/2}}, \qquad (2.6.10)$$

which follow from 2.2.12. For example, it can be shown that the transport in the right-hand boundary layer is $Q_R^* = (g/f)(d^{*2}(w^*/2,-\infty) - D_\infty^2)$, where $d^{*2}(w^*/2,-\infty)$ denotes the right wall depth in the basin. The Bernoulli function on the right wall is $B_R^* = \frac{1}{2}(g/D_\infty)[D_\infty{}^2 + d^{*2}(w^*/2,-\infty)]$. Complete knowledge of the right-wall basin flow therefore requires any two of Q_R^*, B_R^*, $d^*(w^*/2,-\infty)$, or D_∞.

There does not appear to be a simple analytical expression for $Q(w_c,q)$ (though see Exercise 4). Figure 2.6.3 shows some plots of Q as a function of w_c for various values of q. For fixed w_c it is apparent that the transport decreases as q increases and therefore the largest transport occurs for zero potential vorticity. One should exercise caution in interpreting this result, however. If the dimensional critical width w_c^* is held fixed, then fixed w_c means that D and therefore B_R^* is fixed. The maximum in Q for zero q occurs when B_R^* is held fixed. As shown by Iacono (2006), the same is not true when the scaling for D is based on B_L^*. The volume transport then has a maxima at finite values of q.

Exercises

1) By writing (2.5.14) in dimensional form and introducing Δz_R^* as a depth scale, show that the much simpler transport relation (2.6.7b) is obtained.

2) Among the upstream states that are possible at a given location in Figure 2.6.1, show that the limiting case of separated upstream basin flow occurs when $q\,B|_{w/2}$ falls below $q^2 Q + \frac{1}{2}$.

3) *Asymptotic properties of the function* $Q(w_c,q)$.

(a) Using (2.6.3) and the form of (2.6.5a) with $(B_R - h_c) = 1$, show that

$$\lim_{w_c \to 0} Q = \left(\frac{2}{3}\right)^{3/2} w_c\,,$$

and thus the slope of all the Figure 2.6.3 curves at the origin is $(2/3)^{3/2}$ regardless of the value of q.

(b) Next show that for a given q=constant curve in Figure 2.6.3, that separation of the flow at the sill section first occurs where Q=1/2, corresponding to $q = 2T_c^2/(1+T_c^2)$ or

$$w_c = 2q^{1/2} Tanh\left(\frac{q}{2-q}\right)^{1/2}.$$

Note that separation can occurs only for 0<q<1.

Hint: Use the velocity profile (2.2.4) along with the fact that the right-wall value of v is zero when the flow is critical.

(c) Note that the results in (a) and (b) provide endpoints for the curves with $0 \le q \le 1.0$. For $q>1$ show that

$$\lim_{w_c \to \infty} Q = q^{-1}(1 - \tfrac{1}{2}q^{-1}).$$

Hint: Use the same equations as in part (a).

4) *Iacono's (2006) solution.* A closed formula for the constant-potential vorticity transport from a wide reservoir can be obtained if the upstream conditions are chosen in a particular way. Proceed as follows:

a) Evaluate the Bernoulli relation (2.6.2) at the sill, where the flow is assumed to be critical, leading to

$$T_c^{\,2}(\bar{d}_c - q^{-1})^2 + \frac{Q^2}{4T_c^{\,2}\bar{d}_c^{\,2}} + 2q^{-1}\bar{d}_c = 2q^{-1}\Delta z_I\,,$$

where $\Delta z_I = \overline{B} - h_s$.

b) Use the critical condition (2.6.3) to substitute for the second term and rearrange the result, eventually obtaining a quadratic equation for $\overline{d}_c$. Show that the physically meaningful solution to this equation can be written as

$$\overline{d}_c = \frac{3}{4qS_c^{\,2}}\left[-1+\left(1-\frac{8}{9}S_c^4+\frac{16}{9}S_c^2C_c^2q\Delta z\right)^{1/2}\right]$$

where $S_c = Sinh[\tfrac{1}{2}q^{1/2}w]$ and $S_c = Cosh[\tfrac{1}{2}q^{1/2}w]$.

c) Note that with (2.6.3) rearranged to given

$$Q^2 = 4\overline{d}_c^3T_c^2\left[(1-T_c^2)q^{-1}+T_c^2\overline{d}_c\right]_c$$

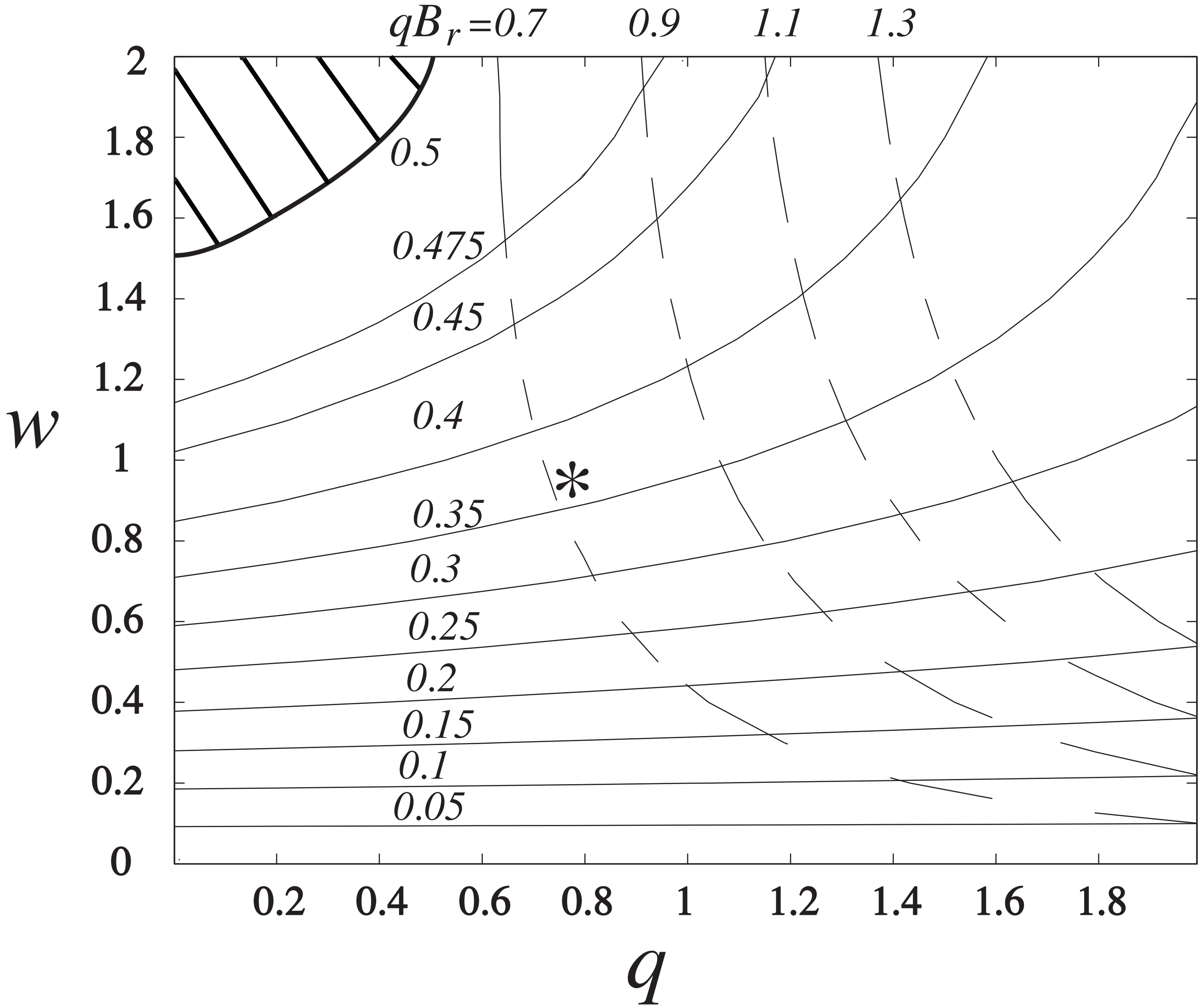
one now has a direct formula for the transport in terms of the new upstream parameter Δz_I. Make some plots of Q *vs* q for fixed Δz_I and fixed sill geometry and thereby show that the maximum flux is obtained at a finite q. Contrast this to the result obtained in Figure (2.6.1).

Figure Captions

Figure 2.6.1 Contours of dimensionless volume flux Q as a function of sill width $w_c=w_c{*}f/(gD)^{1/2}$ and upstream potential vorticity $q=D/D_\infty$. In contrast to the Gill (1977) model, D has been chosen as $g^{-1}B{*}_R - h_c{*}$. The hatched region has separation of the current from the left-hand sill wall. The dashed contours indicate the values of qB_R below which separation occurs from the left wall of the upstream basin. Separation of the sill flow from the left wall occurs within the hatched region; there Q=1/2. The star indicates the values of q and w_c used in Figure 2.6.2. (From Whitehead and Salzig 2001.)

Figure 2.6.2. Profiles of the dimensionless surface elevation $z{*}/D_\infty (= qz)$ in the upstream basin and at the sill, all for q=0.8 and w_c=1. The dimensionless transport Q equals 0.375 in each case. (a) $qB_R = 7$, (b) $qB_R = 1.375$ (c) $qB_R = 1.0$ (d) $qB_R = 0.874$. (From Whitehead and Salzig, 2001.)

Figure 2.6.3 The dimensionless flow rate Q as a function of the dimensionless sill width w_c for various values of q. The q=0 curve merges with the constant value Q=1/2 (see 2.4.15) for separated flow $w_c \geq \sqrt{2}$. The solid lines are exact values, calculated by Whitehead (2005) and equivalent to the data shown in Figure 2.6.1. The starred curve is an approximation based on equation 2.5.33. (From Whitehead and Salzig, 2001.)



qB_r =0.7
0.9
1.1
1.3
0.5
0.475
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
*
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
w
q

Figure 2.6.1

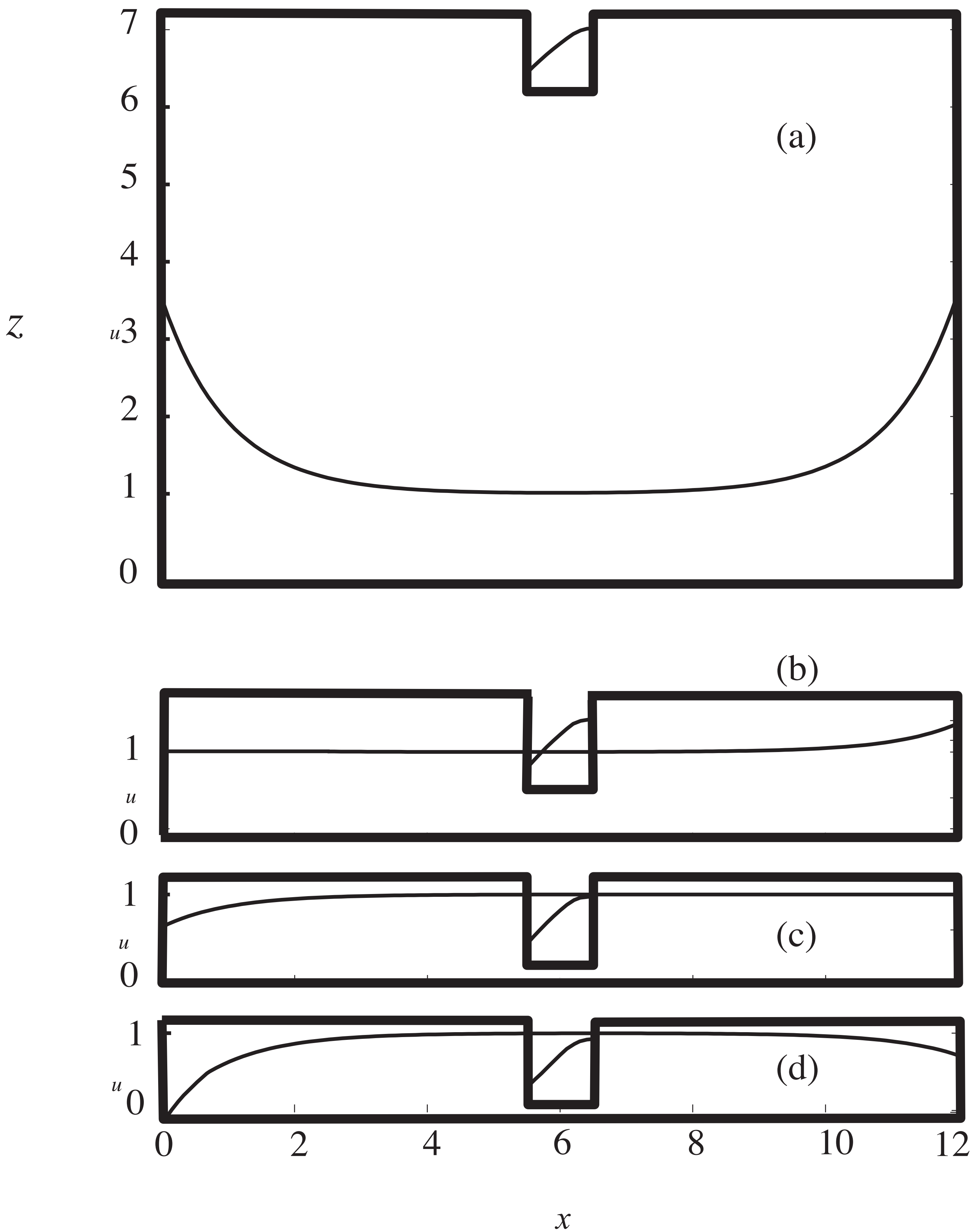
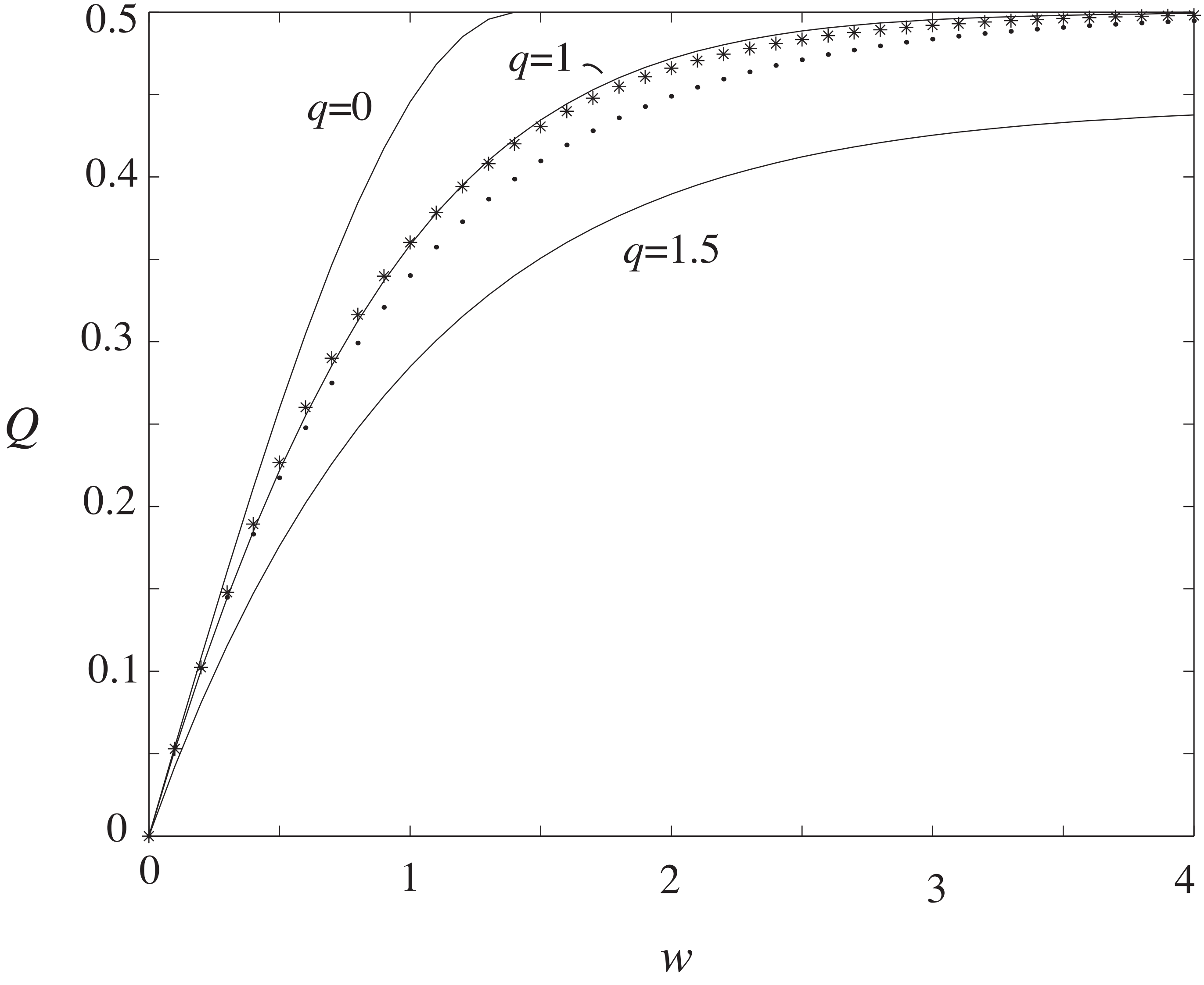


Fig 2.6.2



2.6.3