3.3 Rossby Adjustment in a Channel: Fully Nonlinear Case.

We now come to the most difficult case of the channel Rossby adjustment: that in which the initial depth difference is moderate or large. Let

$$d(x, y, 0) = \begin{cases} 1 & (y < 0) \\ d_o & (y > 0) \end{cases},$$
(3.3.1)

with u(x,y,0)=v(x,y,0) as before, so that $1-d_o$ corresponds to the step height 2a of the previous sections. For O(1) values of $1-d_o$ there is no longer a clear spatial separation between the Kelvin waves and the potential vorticity front. In the limiting case $d_o=0$, the channel is initially dry downstream of the barrier and the leading edge of the advancing intrusion *is* the potential vorticity front. The introduction of a finite step size brings other new processes into play, including the separation of the fluid from the left wall and the formation of shocks. The results described below are largely due to Helfrich et al. (1999).

(a) Case $d_o=0$.

The case of zero depth on the downstream side of the barrier is somewhat easier to treat and will be considered first. Some of the general features of the flow can be anticipated. After the barrier is removed, we expect that the fluid will spill forward into y>0 forming an intrusion (Figure 3.3.1). The linear solution suggests that the velocities along the right wall will be largest and therefore the leading edge or nose of the intrusion should follow that wall. We denote the position and speed of the nose by y_{nose} and c_{nose} . Behind the nose the width of the intrusion will gradually increase until contact with the left wall is made at $y=y_{sep}$. The speed c_{sep} of the separation point is shown to be positive in Figure 3.3.1, and this will be confirmed later.

It is possible to approximate the solution for this case using semigeostrophic theory for uniform potential vorticity, the tools of which were developed in sections 2.2 and 2.3. Central to the semigeostrophic approximation is the assumption of gradual variations in the y-direction, a condition that is clearly violated near y=0 in the early stages of evolution. Nevertheless, there is some hope that, as time progresses, the changes in y will become gradual enough that semigeostrophic theory will capture the predominant features of the solution. We will therefore describe this theory and compare it to numerical solutions based on the full shallow water equations.

The semigeostrophic solution can be obtained using the method of characteristics in much the same way that the nonrotating dam-break problem was solved (see Section 1.3). Where the flow remains attached, the independent variables v and d are replaced by

 \overline{d} and \hat{d} , one-half the sum and difference of the wall depths (see 2.2.5,6). The initial conditions are therefore

$$\overline{d}(y,0) = \begin{cases} 1 & (y<0) \\ 0 & (y>0) \end{cases} \text{ and } \hat{d}(y,0) = 0 \tag{3.3.2}$$

Although the characteristic speeds c_{\pm} and Riemann invariants R_{\pm} are more complicated than in the nonrotating case, the central ideas involved in the construction of the characteristic curves remain the same. In particular, one must address the fact that there is an infinity of solutions that satisfy the discontinuous initial condition. Replacement of the discontinuity with a short interval of continuous depth change (as in Figure 1.3.2) resolves this problem. As was the case in the nonrotating version, the assumption of uniform R_{\pm} within this interval leads to the unsatisfactory conclusion that the leading edge of the intrusion will move towards *negative y*. We therefore take R_{\pm} as being uniform as before. It follows from (2.2.23-24) that

$$R_{+} = T^{-1}\hat{d} + \int^{\overline{d}} \alpha^{-1/2} \left[1 - T^{2}(1 - \alpha)\right]^{1/2} d\alpha$$

$$= T^{-1}\hat{d} + \overline{d}^{1/2} \left[1 - T^{2}(1 - \overline{d})\right]^{1/2} + \frac{1 - T^{2}}{T} \log\left\{2\overline{d}^{1/2}T + 2\left[1 - T^{2}(1 - \overline{d})\right]^{1/2}\right\} (3.3.3)$$

$$= 1 + \frac{1 - T^{2}}{T} \log(2T + 2),$$

where $T = \tanh(w/2)$. The first two lines follow from the definition of R_+ for nonseparated flow. Also the value q of the potential vorticity has been set to its initial value of unity. The final step results from the evaluation of R_+ using the initial conditions (3.3.2). The result is a relationship between \overline{d} and \hat{d} that holds for $y < y_{sep}$.

In accordance with the ideas developed in Section 1.3, the uniformity of R_+ over the whole body of fluid implies that the '-' characteristic curves must have constant slope. The characteristic speeds are given by (2.2.22)

$$c_{-} = T^{-1}\overline{d} - \overline{d}^{1/2} \left[1 - T^{2} (1 - \overline{d}) \right]^{1/2} \qquad (y < y_{sep})$$
(3.3.4)

Calculation of the values of c_{-} over the short interval replacing the step gives a value that increases from left to right (see Exercise 1), and thus the '-' characteristic curves fan out from the origin, as in Figure (1.3.2). Conservation of $R_{-}(\overline{d}, \widehat{d})$ along each of the fanning curves in conjunction with the independent relationship (3.3.4) implies that \overline{d} and \widehat{d} are individually conserved along each curve (just as v and d were conserved along the '-' characteristics of the nonrotating problem).

In the separated portion of the flow, \overline{d} and \hat{d} are no longer independent. A convenient choice of variables is \overline{d} and the separated width w_e of the current. As shown

in Section 2.3 the definition (3.3.4) of c_1 remains valid provided we replace \hat{d} by \overline{d} and w by w_e . Thus

$$c_{-} = T_{e}^{-1} \overline{d} - \overline{d}^{1/2} \left[1 - T_{e}^{2} (1 - \overline{d}) \right]^{1/2} \qquad (y_{sep} < y < y_{nose}) \quad (3.3.5)$$

where $T_e = \tanh(w_e/2)$. An analytical expression for $R_+(\overline{d}, w_e)$ is not known and values must therefore be tabulated. The procedure for doing so [Stern et al. (1982), Kubokawa and Hanawa (1984), or Helfrich et al. (1999)] is algebraically involved and will not be detailed here. Nonetheless, the procedure for obtaining the solution in the separated region is essentially the same as in the attached region. We set R_+ equal to its initial value (the final expression in 3.3.3) and thereby obtain an implied relationship between \overline{d} and w_e that must hold throughout the separated region. The values of \overline{d} and w_e along each '-' characteristic are then determined from this relationship in conjunction with (3.3.5).

Solutions to the unapproximated initial-value problem require a numerical algorithm for the full shallow water equations. The finite-difference code in the present case is summarized by Helfrich et al. 1999. It allows the fluid to vanish¹, resulting in the formation of a free edge, either in the interior or through side-wall separation. The code also allows for the formation and maintanance of jumps and bores consisting of large changes in the depth and velocity over a few grid points. We have already shown that ideal hydraulic jumps in one-dimensional flows conserve volume flux and flow force. These ideas extend to two-dimensional jumps in rotating systems in a way that will be made clear in Section 3.5. The numerical code in question is written in a way that inforces conservation of the correct properties and, in particular, adds no mass or momentum to the flow. A key step is to base the code on the flux form (2.1.17) of the shallow water equations, which is essentially a conservation law for the quantities that need to be conserved across the jump.

The semigeostrophic solution (upper panel of Figure 3.3.2) can be compared with numerical solutions to the full 2-D shallow water equations (lower panel) for the case w=2.0 (a channel width equal to two deformation radii based on the initial upstream depth). Both solutions agree qualitatively with each other and with the anticipated scenario. For example the separated intrusion becomes increasingly narrow with time, in agreement with the fanning characteristic curves. The semigeostrophic theory does not capture the oscillations that are apparent in the lower panel of the figure and are associated with Poincare' waves. These waves are filtered by the semigeostrophic approximation. In the full numerical solution for w=4 (Figure 3.3.3), the rarefying nature of the intrusion is apparent and the oscillations are more pronounced.

The value of c_{nose} can be determined from the semigeostrophic solution by noting that the nose moves along a '-' characteristic. Taking the simultaneous limit $\overline{d} \to 0$ and

¹ The fluid depth is formally judged to vanish when d reaches a value of the order 10⁻⁵. A thin sheet of fluid with this thickness is then maintained over those portions of the channel considered 'dry'.

 $w_e \rightarrow 0$ in (3.3.5) leads to $c_{nose} \rightarrow 2\overline{d} / w_e$. The values of \overline{d} and w_e are also constrained by the requirement that $R_+(\overline{d}, w_e)$ is equal to its initial value, as specified by (3.3.3). However, $R_+(\overline{d}, w_e)$ is only known in tabulated form, and the limit of $2\overline{d} / w_e$ as \overline{d} and w_e both approach zero must be ascertained numerically. The resulting c_{nose} depends on the value of T (Figure 3.3.4). The nose speed increases from its nonrotating value of 2 (dimensionally $2\sqrt{gD}$, where D is the initial depth behind the barrier) for w=0 to the asymptotic value 3.8 for infinite w (or T=1). The circles indicate the nose speed as given by the full numerical solution. Although the 'exact' result mirrors the behavior of the semigeostrophic result, the actual values are substantially less than those predicted by semigeostrophic theory.

The value of c_{sep} can be predicted by setting $c = c_{sep}$ and $\overline{d} = \hat{d}$ in (3.3.4), giving a relation between c_{sep} and the value of \overline{d} at the separation point. A second relation for this \overline{d} can be obtained by setting $\overline{d} = \hat{d}$ in (3.3.3). The relationship between c_{sep} and T(w) is then obtained through elimination of \overline{d} between these two relations, a procedure that must be carried out numerically. The prediction (lower solid curve in Figure 3.3.4) agrees well with the c_{sep} predicted by the full solution (squares). In the narrow channel limit $T \rightarrow 0$ the nonrotating limit of equal c_{sep} and c_{nose} is obtained, implying that the free edge is directed perpendicular to the channel walls. In the other extreme $(T \rightarrow 1)$, c_{sep} vanishes implying that the separation point remains pinned to the position of the initial barrier.

As in the nonrotating version of the dam break problem, the steady state achieved for $t\rightarrow\infty$ is hydraulically critical. The corresponding values of \overline{d} and \hat{d} may be found by setting c=0 in 3.3.4. A second relationship between these \overline{d} and \hat{d} is provided by (3.3.3). The volume transport $2\overline{d}\hat{d}$ of the resulting flow, shown by the solid curve in Figure 3.3.5, agrees quite well with the numerically-determined values (diamonds). Also shown is the nonrotating transport $(2/3)^3 w$ (the dashed line) and the transport observed in the numerical simulations (diamonds). Both approach the nonrotating prediction of $(2/3)^3 w$ (dashed line) obtained in Section 1.3. For large w the predicted and observed transports approach the asymptotic value 0.5 (dimensionally $0.5gD^2/f$).

When the full dam break problem is carried out in the presence of a deep, overlying or underlying fluid, the behavior of the nose region is quite different. As discussed on Section 4.5, the leading edge becomes blunt and propagates at a much slower speed than that predicted above.

(b) The case $d_0 > 0$.

When the initial depth ahead of the barrier is nonzero, the flow will differ in many respects from what has just been discussed, or so one might anticipate. Separation of the fluid from the left wall will presumably not occur. The potential vorticity front formed between the initially shallow and deep fluid will be distinct from the leading edge of the Kelvin wave. Most importantly, the Kelvin wave may break and form a shock. As will be

shown later in the chapter, potential vorticity is generally not conserved across such a shock and the foregoing theory, which was based on the assumption of globally uniform q, becomes more difficult to justify. All of these factors contribute to an understanding of the partial dam break problem that is relatively poor. We will simply present some numerical solutions that show the major features of the flow.

Based on simulations, there appear to be two main forms that the Kelvin wave shock may take. The first is favored by small values of d_0 or large w. An example for the case $d_0=0.1$ and w=4 (Figure 3.3.6) shows a fully developed shock near y=15 at t=12. The leading edge consists of an abrupt change in depth, the amplitude of which decays away from the right wall. Virtually none of the disturbance is felt at the left wall. In addition, the leading edge curves backwards, forming an oblique angle with the right wall. The potential vorticity front, which is shown as a heavy, solid line, lies just to the rear of the shock along right wall. As one moves away from the right wall the position of the front lags further behind the shock. At the left wall, the front remains pinned near y=0.

The second type of shock is reminiscent of the unstable Kelvin wave discussed in Section 3.2. It is favored by small or moderate $1-d_o$ or w. In an example based on $d_o=.25$ and w=1 (Figure 3.3.7) the potential vorticity front now lags well behind the shock at all x. In addition, the entire front now moves forward and is no longer pinned to the left wall at y=0. The shock, which can be seen at t=20 near y=20 is now felt across the entire channel width. The Poincaré waves just behind the front and along the left-wall may have been generated resonantly by the Kelvin wave, as described in Sec 3.2b. This generation mechanism is further suggested by the lack of similar oscillations near the backward-propagating, left-wall Kelvin wave (which is rarefying and which roughly occupies the interval -20 < y < -5 at t=20). Both of the shocks described above will be revisited later in this chapter.

The transport Q of the final steady state, measured at y=0, shows that larger w and smaller d_0 lead to larger values of Q (Figure 3.3.8a). For comparison, the transport predicted for the non-rotating version of the dam break (Stoker, 1957) in a channel of unit width is indicated by a solid line. In addition the 'geostrophic control' bound $(1-d_0^2)/2$ based on the initial depth is shown as a dashed line. The calculated transport exceeds this bound only once, and then only slightly. However, it should be reemphasized that the loss of the crossing flow for all but large w obscures the interpretation of this result.

The increase in Q with w for any fixed d_o is similar to that of the semigeostrophic transport for $d_o=0$ (solid line in Figure 3.3.5). When the measured Q values are normalized by the latter using common values of w, the data points for each d_o nearly collapse (Figure 3.3.8b). The solid curve in the same figure is the Stoker (1957) prediction for zero rotation, scaled as above.

Exercises

1. Prove that the intersection of the solid line with the *Q* axis is Figure 3.3.8 lies at $Q = \left(\frac{2}{3}\right)^3 = .295...$ (Hint: You will need to use a result from Section 1.3.)

Figure Captions

3.3.1 Schematic plan view of the intrusion following a dam break in a rotating channel.

3.3.2 Contours of the depth *d* field at t=10 for a channel with w=2.0. (*a*) The semigeostrophic solution. (*b*) The numerical solution to the full shallow water equations. (Helfrich et al. 1999, Figure 6)

3.3.3. Full numerical solution for d(x,y,t) at the indicated times for w=4.0. The contour interval is 0.1. (Helfrich et al. 1999 Figure 7)

3.3.4 The intrusion nose speed c_{nose} and the separation point speed c_{sep} as functions of the width parameter *T*. The solid curve corresponds to theory and the symbols correspond to the numerical model results. (Based on Helfrich et al. 1999 Figure 3)

3.3.5 Final transport Q vs. w for the full dam break ($d_0=0$). The solid curve is from the semigeostrophic theory; it is also used as a scale factor in Figure 3.3.8b, where it is referred to as Q_0 . The dashed line is based on the nonrotating theory and the diamonds correspond to the numerical solution. (Helfrich et al. 1999, Figure 9)

3.3.6 Full numerical solution for d(x,y,t) to the partial dam break problem with w=4.0 and $d_0=0.1$. The contour interval is 0.05. (Helfrich et al. 1999, Figure 15)

3.3.7 Same as Figure 3.3.6, except that w=1 and $d_0=0.5$. (Helfrich et al. 1999, Figure 14)

3.3.8 The final transport Q as a function of d_o and w. In (*a*) the solid line indicates the nonrotating theory for a channel of unit width, while the dashed line is the geostrophic transport $(1-d_o^2)/2$ based on the initial difference in depths. In (*b*) the transport is normalized by the transport Q_o from the semigeostrophic theory for $d_o=0$. In both (*a*) and (*b*) w=0 (\bigcirc), 0.2 (\square), 0.5 (\diamondsuit), 1.0 (\bigtriangledown), 2.0 (\precsim), 4.0 (\triangle).





w=2.0



w=4.0

Fig. 3.3.3





Fig. 3.3.5



w=4.0



Fig. 3.3.7





Fig. 3.3.8