

### 3.9 Ageostrophic Instability

Our discussion to this point has largely avoided the question of stability. In fact, nearly all of the internal flows under discussion are unstable in some respect. The presence of the horizontal velocity discontinuity between the moving layer and the overlying fluid gives rise to interfacial instabilities. In a single-layer (reduced gravity) system, the length scale of unstable disturbances is finite and the disturbance pressure is therefore non-hydrostatic. As the magnitude of the velocity discontinuity decreases, the wave lengths of unstable waves also decreases. The instabilities are avoided in traditional shallow-water models with single layers because of the limitation to long wavelengths and the consequent hydrostatic approximation. It is natural to ask, however, whether the presence of the instabilities, and the mixing that they can cause, will wreck the idealization of the moving fluid as a single layer with uniform density. In cases where this length scale is small compared to the fluid depth, the instability may result in overturning and mixing that is limited to the vicinity of the interface. The sharp interface is replaced by a transition layer that may remain thin compared to the layer depth. The single-layer, reduced-gravity idealization may then still be appropriate long-wave behavior. More on this point will follow in Chapter 5.

It is also reasonable to expect rotating-channel flows to be subject to instabilities that effect the horizontal structure. These include the well-documented barotropic instabilities that can arise in the presence of horizontal variations in velocity, and baroclinic instabilities that arise in rotating flows with horizontal variations in potential energy. Oceanic and atmospheric jets, boundary currents, and broad scale circulations are all subject to these instabilities. The theory for this subject has been developed most thoroughly within the quasigeostrophic approximation (e.g. Pedlosky, 1987). Hydraulically driven, rotating flows typically have strong horizontal shear and large variations in potential energy (interface elevation), and would therefore appear to be particularly vulnerable to such instabilities. Certain outflow plumes from the Mediterranean and the Denmark Strait are known to contain horizontal eddies that span with stream width and that could be attributed to these instabilities. These flows are non-quasigeostrophic and a stability analysis requires that one abandon this approximation by allowing the horizontal velocity to be ageostrophic and the layer thickness to vary by large amounts across the flow, possibly vanishing at the edges.

At the time of this writing, the intersection between rotating hydraulics and ageostrophic instability is unclear. For example, the extent to which the steady flows of the Whitehead et al. (1974) and Gill (1977) models are unstable is not known. Nor is it understood how the presence of instabilities might alter these flows. For example, it is possible that the instabilities might act only in the supercritical portions of the flow and therefore have no upstream effects. Our inclusion of ageostrophic instability analysis is therefore made in hope that other investigators will use the basic tools to answer some of these questions. Though it is not strictly necessary, the reader will benefit from some rudimentary knowledge of instability theory (e.g. Chapter 7 of Pedlosky 1986). The

following development is based largely on the work of Griffiths, Killworth and Stern (1982), Ripa (1983), and Hayashi and Young (1987), with some generalization in the bottom topography used by these authors.

a) *Remarks on the stability problem and review of standard conditions for instability.*

We will be concerned with *linear* stability; that is the stability of a *basic* state to infinitesimal disturbances. *Instability* means that it is possible to find an infinitesimal disturbance that will grow in time and lead to a permanent, finite departure from the basic state. We will also confine our discussion to flows that are inviscid and unforced, and therefore preserve their total energy and momentum. The growth of an unstable disturbance to the basic state must then occur without the benefit of any external forcing or dissipation. There certainly are classes of instabilities that act in non-conservative flows and that owe their existence to the presence of friction, but these will not be considered here.

In the traditional analysis of the barotropic stability of large-scale ocean currents and atmospheric winds, the basic state is parallel and zonal: nondimensionally  $u=U(y)$ . If the basic state has constant depth and takes place on an  $f$ -plane, stability is informed by Rayleigh's (1880) inflection point theorem. In particular,  $d^2U/dy^2$  must change sign at some value of  $y$  for instability to be possible. Kuo (1949) showed that the  $\beta$ -plane extension of this result is that the potential vorticity gradient  $\beta - d^2U/dy^2$  change sign. Charney and Stern (1962) extended this result further to include quasigeostrophic flows with continuous stratification. Instability requires that the horizontal gradient of potential vorticity (including the boundary contribution) must change sign at some point within the cross section. For a single layer with reduced gravity dynamics, this means that  $\beta - d^2U/dy^2 + f^2U/(gD)$  must change sign. (Lipps, 1963).

The above necessary conditions can be strengthened by a result due to Fj rtorft (1950). His sufficient condition for stability of a barotropic flow is satisfied if a constant  $\alpha$  can be found such that  $(U - \alpha)(\beta - d^2U/dy^2) \leq 0$  for all  $y$ .<sup>1</sup> As an example, consider a 2D shear flow with  $\beta=0$  and suppose that  $d^2U/dy^2$  changes sign at  $y^*=y_o$ . Rayleigh's inflection point theorem is therefore satisfied and the flow may be unstable. However, stability may still be demonstrated by choosing  $\alpha=U(y_o)$ , so that the Fj rtorft sufficient condition for stability becomes  $(U(y)-U(y_o))(d^2U/dy^2) \geq 0$  for all  $y$  in the domain of interest. If it happens that the profile is such that  $U(y)-U(y_o)$  and  $d^2U/dy^2$  have the same sign, then the flow is stable. Fj rtorft's theorem is closely related to a sufficient condition for stability, developed below, that applies to shallow-water flows.

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<sup>1</sup> If the basic potential vorticity  $\bar{q} = \beta - d^2U/dy^2$  is considered to be a function of the streamfunction,  $\bar{q} = \bar{q}(\psi)$ , then Fj rtorft's condition for stability is satisfied if a frame of reference, moving with constant speed  $c$ , can be found such that  $d\bar{q}/d\psi \geq 0$ .

In keeping with our convention for a rotating channel, we consider a steady basic state  $v = \mathcal{V}(x)$  and  $d = \mathcal{D}(x)$ , whose stability is to be examined. The basic flow is parallel, and therefore in geostrophic balance, and the channel cross section is arbitrary but uniform in  $y$  (Figure 3.9.1a). The channel may contain vertical sidewalls  $x = \pm w/2$ , or the depth may vanish at one or both edges:  $x = -a(y, t)$  and  $x = b(y, t)$ .

*b. Energy and Momentum in an unstable wave.*

Instability is traditionally defined and measured in terms of the growth in time of some positive definite quantity, usually a wave energy norm<sup>2</sup>. The wave draws on energy available in the mean ( $y$ -average) state due to horizontal shear or to gradients in the elevation of the upper interface. As the wave energy grows, the energy associated with the mean diminishes. For the shallow water models used in hydraulics, in which Poincaré and Kelvin waves, and their relatives, are permitted, the energy associated with the wave is no longer positive definite. The notion that the wave draws energy from the mean flow must be reexamined. The sufficient conditions for quasigeostrophic stability are no longer sufficient; in fact, the instabilities that are most interesting from an energy perspective can occur when the potential vorticity gradient is zero.

The dimensionless shallow-water energy equation is obtained from  $u d \times (2.1.5) + v d \times (2.1.6) + d \times (2.1.7)$ :

$$\frac{\partial}{\partial t} \frac{d(u^2 + v^2) + d^2 + 2dh}{2} = -\nabla \cdot (\mathbf{u} dB). \quad (3.9.1)$$

The scaling introduced in Section 2.1, with  $\delta=1$ , is in effect and the Bernoulli function  $B$  therefore takes its full two-dimensional form  $\frac{1}{2}(u^2 + v^2) + d + h$ . Suppose that the disturbed flow is periodic in  $y$ , or that the disturbance is isolated in  $y$ . Let  $A$  represent the horizontal region occupied by the fluid, the wetted area, over one wavelength. Integration of (3.9.1) over  $A$  and use of the side edge condition  $u d = 0$ , valid for vertical walls or for a free edge with vanishing depth, then yields

$$dE/dt = 0$$

where

$$E = \frac{1}{2} \iint_A [d(u^2 + v^2) + d^2 + 2dh] d\sigma \quad (3.9.2)$$

and  $d\sigma$  is the elemental area.

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<sup>2</sup> Other norms are used, including enstrophy.

The total momentum of the flow over one period is also conserved, as can be shown from the following form of the momentum flux equation (see Exercise 1):

$$\frac{\partial}{\partial t}[d(v+x)] + \nabla \cdot [(v+x)\mathbf{v}d] + \frac{\partial}{\partial y}\left(\frac{d^2}{2}\right) = -d\frac{\partial h}{\partial y}. \quad (3.9.3)$$

With  $dh/dy=0$ , integration over  $A$  yields

$$dM/dt=0,$$

where

$$M = \iint_A d(v+x)d\sigma. \quad (3.9.4)$$

We now separate the flow into a basic part ( $\mathcal{V}$ ,  $\mathcal{D}$ ,  $\mathcal{A}$ ) and a small perturbation. The amplitude of the perturbation is measured by the dimensionless parameter  $\varepsilon \ll 1$ . The flow field is formally represented as

$$\begin{aligned} v &= \mathcal{V} + \varepsilon v' + \varepsilon^2 v'' + \dots, \\ u &= \varepsilon u' + \varepsilon^2 u'' + \dots \\ d &= \mathcal{D} + \varepsilon d' + \varepsilon^2 d'' + \dots \\ A &= \mathcal{A} + \varepsilon A' + \varepsilon^2 A'' + \dots \\ q &= Q + \varepsilon q' + \varepsilon^2 q'' + \dots \end{aligned} \quad (3.9.5)$$

The area perturbation  $\varepsilon A' + \dots$  is due to lateral displacements of the free edges of the current and is zero when the fluid is bounded on both sides by vertical walls. If the edges are free, however, changes in the edge positions alter the horizontal area over which the flow exists (Figure 3.9.1b).

Linear instability analysis determines the lowest order perturbation quantities like  $v'$ ,  $d'$ , etc., which generally have a wave-like structure in  $y$ . We will refer to these lowest order quantities collectively as the *wave* field. The wave field can be considered as having no mean with respect to  $y$ . Such a mean can be shown to be time-independent and can therefore be disposed by redefining the basic flow. The entire perturbation field:  $\varepsilon v' + \varepsilon^2 v'' + \dots$ ,  $+\varepsilon d' + \varepsilon^2 d'' + \dots$ , etc. will be referred to as the *disturbance*. The higher order contributions to the disturbance field, starting with  $\varepsilon^2 v''$ , etc., may have time-varying means with respect to  $y$ . Thus, if  $\bar{v}$  represents the average of  $v$  over a spatial period in  $y$ , then

$$\bar{v} = \mathcal{V} + \varepsilon^2 \bar{v}^{(1)} + \dots.$$

To the extent that higher order terms can be neglected, the total energy and momentum can now be decomposed into distinct parts associated with the wave and the mean. The latter can further be expressed as a sum of the basic state energy and the energy due to the mean of the disturbance. Substitution of the partitioned fields into the definitions of  $E$  and  $M$ , and neglect of  $O(\varepsilon^3)$  terms leads to

$$E = E_b + E_w + E_m$$

where

$$E_b = \frac{1}{2} \iint_{\mathcal{A}} [\mathcal{D}V^2 + \mathcal{D}^2 + 2\mathcal{D}h] d\sigma,$$

$$E_w = \frac{\varepsilon^2}{2} \iint_{\mathcal{A}} [\mathcal{D}(u'^2 + v'^2) + 2Vv'd' + d'^2] d\sigma,$$

and

$$E_m = \frac{\varepsilon^2}{2} \iint_{\mathcal{A}} [2\mathcal{D}Vv'' + (V^2 + 2\mathcal{D} + 2h)d''] d\sigma.$$

To avoid some unnecessary complexity we have temporarily assumed that the flow is bounded by rigid channel walls, and thus the disturbed area  $\varepsilon A' + \dots$  is zero. If the effect of free edges is included, a second expression involving an integral over  $\varepsilon A'$  is added to the final integral  $E_m$  (See Hayashi and Young, 1987 for more details.)

The term  $E_b$  above is just the energy associated with the basic flow. The quantity  $E_w$ , sometimes called the *wave energy*, is the energy associated with the quadratic terms in the perturbation fields. The wave energy can be calculated from the solution to the linearized problem for  $u'$ ,  $v'$ , etc. In two-dimensional or quasigeostrophic flow, the contribution to  $E_w$  from term involving  $Vv'd'$  is absent due to the fact that the depth perturbation is either zero or negligibly small. In this case  $E_w$  consists of a sum of non-negative terms and is used as a measure of the size or growth of the perturbation. In the present shallow-water setting, the term  $Vv'd'$ , and possibly the entire wave energy, can be negative. Finally, the term  $E_m$  is the contribution to the energy from the mean of the disturbance. The first order perturbations have no mean and thus  $E_m$  is composed of contributions from the means of the  $O(\varepsilon^2)$  fields  $v''$  and  $d''$ . The individual constituents cannot be calculated from the linearized problem, though as we will later see, the complete sum  $E_m$  can be.

For momentum,

$$M = M_b + M_w + M_m$$

where

$$M_b = \iint_{\mathcal{A}} \mathcal{D}(V + x) d\sigma,$$

$$M_w = \varepsilon^2 \iint_{\mathcal{A}} (v'd') d\sigma,$$

$$M_m = \varepsilon^2 \iint_{\mathcal{A}} [d''(\mathcal{V} + x) + \mathcal{D}v''] d\sigma ,$$

again neglecting terms of  $O(\varepsilon^3)$  and assuming vertical side walls.

Another quantity of significance for stability analysis is the *disturbance energy*, defined as

$$E_d = E_m + E_w .$$

It is the sum of the wave energy and the energy associated with changes in the mean fields. It is also the difference  $E - E_b$  between the energy of the actual flow and that of the basic flow. Since the total energy  $E$  is conserved,  $E_d$  is also conserved. The disturbance energy of a growing wave that has sprung from an infinitesimal instability is zero. One way to think about this is to consider a disturbance observed to have finite but small amplitude of  $O(\varepsilon)$ . The individual terms that constitute  $E_d$  are  $O(\varepsilon^2)$  and an uninformed observer might guess that  $E_d$  is also  $O(\varepsilon^2)$ . In fact, the disturbance can be traced back in time to when its amplitude is smaller. By retreating further in time, the disturbance amplitude, and therefore  $E_d$ , can be made arbitrarily small. The conserved disturbance energy is therefore essentially zero. The same remarks apply to the *disturbance momentum*, defined by  $M_d = M_m + M_w$ .

If, on the other hand, the observed disturbance has non-zero energy (or momentum) then it is clear that the disturbance, or some portion thereof, cannot have evolved as the result of an infinitesimal instability. A flow for which all possible disturbances alter the energy is therefore stable to infinitesimal perturbations.

A simple demonstration of the principle of zero disturbance energy for an unstable system can be made with a pendulum (Figure 3.9.2). First consider its stable equilibrium, with the arm and weight hanging straight down. A moderate perturbation sets the weight in periodic motion. Let  $a$  denote the maximum vertical displacement, relative to its equilibrium position, that the weight achieves during its swaying motion (frame A of the figure). The energy associated with the swaying motion is then proportional to  $a^2$ . This is also the disturbance energy: the difference between the total energy of the pendulum and its basic state energy. Note that all possible disturbances add energy to the system relative to the basic state.

Next consider the unstable equilibrium state, with the weight and arm suspended straight up (Figure 3.9.1+1B). A slight nudge sets the pendulum in motion and we consider a snap shot of that motion when the weight has undergone the same vertical displacement  $a$  as before. The total energy of the system at this point is the same as the basic state energy, or at least can be made to approach the basic energy by making the initial 'nudge' infinitesimally small. The disturbance energy is therefore essentially zero.

Another quantity of importance is the lateral displacement  $x-x_0 = \delta(y,t;y_0)$  of a fluid column away from its original position  $x_0$  in the background state. Thus

$$\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial y} \right) \delta = u \quad \text{or, in linearized form,}$$

$$\left( \frac{\partial}{\partial t} + \mathcal{V}(x_0) \frac{\partial}{\partial y} \right) \delta = u'(x_0, y, t). \quad (3.9.6)$$

### c) Ripa's Theorem

A sufficient condition for stability (Ripa, 1983) can be formulated by making bounds based on the conservation laws for disturbance energy  $E_w + E_m$  and momentum  $M_w + M_m$ . The 'wave' constituents  $E_w$  and  $M_w$  are composed largely of bound-friendly quadratic terms like  $v'^2$ . The terms that contribute to  $E_m$  and  $M_m$  are less so and require a bit more analysis. To this end we consider the linearized shallow water equations for the disturbance fields:

$$\left( \frac{\partial}{\partial t} + \mathcal{V} \frac{\partial}{\partial y} \right) u' - v' = - \frac{\partial d'}{\partial x} \quad (3.9.7a)$$

$$\left( \frac{\partial}{\partial t} + \mathcal{V} \frac{\partial}{\partial y} \right) v' + Q \mathcal{D} u' = - \frac{\partial d'}{\partial y}, \quad (3.9.7b)$$

$$\left( \frac{\partial}{\partial t} + \mathcal{V} \frac{\partial}{\partial y} \right) d' + \frac{\partial(\mathcal{D} u')}{\partial x} + \frac{\partial(\mathcal{D} v')}{\partial y} = 0, \quad (3.9.7c)$$

and

$$\left( \frac{\partial}{\partial t} + \mathcal{V} \frac{\partial}{\partial y} \right) q' + u' \frac{\partial Q}{\partial x} = 0, \quad (3.9.7d)$$

obtained through substitution of (3.9.5) into the unforced versions of (2.1.5-8) and neglect of  $O(\epsilon^2)$  terms. Here

$$Q = \frac{1 + \partial \mathcal{V} / \partial x}{\mathcal{D}}$$

is the basic state potential vorticity and

$$q' = \mathcal{D}^{-1} \left( \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - Q d' \right)$$

is the perturbation potential vorticity.

It can be shown (see Exercise 2) that the above set leads to

$$\frac{\partial \overline{e_w}}{\partial t} - \mathcal{V} \mathcal{D}^2 \overline{u'q'} = -\frac{\partial}{\partial x} (\mathcal{V} \mathcal{D} \overline{u'v'} + \mathcal{D} \overline{u'd'}) \quad (3.9.8)$$

$$\frac{\partial \overline{m_w}}{\partial t} - \mathcal{D}^2 \overline{u'q'} = -\frac{\partial}{\partial x} (\mathcal{D} \overline{u'v'}) \quad (3.9.9)$$

where

$$e_w = \frac{1}{2} \mathcal{D} (u'^2 + v'^2) + \mathcal{V} v' d' + \frac{1}{2} d'^2 \quad (3.9.10)$$

and

$$m_w = v' d' \quad (3.9.11)$$

are the densities of the wave energy and wave momentum and an overbar denotes an average in  $y$  over a spatial period.

Using the expression (3.9.6) for the linearized particle excursions, it follows from (3.9.7d) that

$$\left( \frac{\partial}{\partial t} + \mathcal{V} \frac{\partial}{\partial y} \right) \left[ q' + \delta \frac{\partial Q}{\partial x} \right] = 0. \quad (3.9.12)$$

The general solution to (3.9.12) can be written

$$q' = -\delta \frac{\partial Q}{\partial x} + F(x, y - \mathcal{V}(x)t). \quad (3.9.13)$$

The first term on the right-hand side is the potential vorticity perturbation due to the transverse displacement of a fluid column in the basic state. It therefore results from a conservative rearrangement of the basic potential vorticity  $Q$ . The second term reflects perturbations in  $q$  due to changes in the potential vorticity of fluid columns from their base values. These changes require some sort of external forcing. As shown by (3.9.13) the  $q'$  anomalies that result are passively advected by the basic velocity. If its initial spatial distribution is arranged advantageously, an isolated anomaly may temporarily amplify as a result of differential advection. According to linear theory, the disturbance will eventually decay, but its temporary growth might in practice lead to nonlinear effects that cause irreversible changes in the flow. The reader is referred to Farrell and Ioannou



(1996) and references contained therein for further insight. The process described does not, however, qualify as instability according to our strict requirement that the disturbance is unforced.

If the forced contribution  $F$  to (3.9.13) is ignored, it follows that

$$\overline{u'q'} = -\frac{\partial}{\partial t} \frac{\delta^2}{2} \frac{\partial Q}{\partial x}.$$

The perturbation potential vorticity flux is therefore due to a motion, on average, of the fluid columns down the gradient of background potential vorticity. If the equations for wave momentum and energy density are integrated across the channel, and the above expression for  $\overline{u'q'}$  is used, one finds

$$\frac{\partial}{\partial t} \iint_{\mathcal{A}} \left( e_w + \mathcal{D}^2 \mathcal{V} \frac{\delta^2}{2} \frac{\partial Q}{\partial x} \right) d\sigma = 0 \quad (3.9.14a)$$

$$\frac{\partial}{\partial t} \iint_{\mathcal{A}} \left( m_w + \mathcal{D}^2 \frac{\delta^2}{2} \frac{\partial Q}{\partial x} \right) d\sigma = 0 \quad (3.9.14b)$$

Comparison with the earlier energy decompositions suggests that the conserved integrals are the disturbance energy  $E_d$  and disturbance momentum  $M_d$ . Consequently, the integrals of the terms involving  $\delta^2$  are the mean energy and momentum, at least within a constant. If the potential vorticity gradient is zero, then the mean energy is identically zero and the disturbance energy  $E_d$  equals the wave energy  $E_w$ . A similar result holds for the disturbance and wave momentum. Instability is still possible as the growth in positive terms such as  $Du^2/2$  is compensated by the potentially negative term  $\mathcal{V}v'd'$  in the wave energy. There is no exchange of energy between the growing wave and the mean flow. The mean flow may change, but the energy associated with that change is zero.

A sufficient condition for stability (Ripa, 1983) can be formulated as follows. Although  $e_w$  is not sign definite, it can be shown (Exercise 3) to be non-negative provided that  $\mathcal{V}^2/\mathcal{D} \leq 1$ , for all  $y$ . That is, a flow for which the local Froude number, dimensionally  $\mathcal{V}^{*2}/g\mathcal{D}^{*2}$ , is everywhere  $\leq 1$ , has non-negative  $e_w$ . More generally, it can be shown that if a constant  $\alpha$  can be found such that  $(\mathcal{V} - \alpha)^2 \leq \mathcal{D}$ , or

$$-\mathcal{D}^{1/2} \leq \mathcal{V} - \alpha \leq \mathcal{D}^{1/2}, \quad (3.9.15a)$$

then  $e_w - \alpha m_w \geq 0$ . With this result in hand, we subtract the product of  $\alpha$  and (3.9.14b) from (3.9.14a). A time integration of the result yields

$$\iint_{\mathcal{A}} \left( (e_w - \alpha m_w) + \mathcal{D}^2 (\mathcal{V} - \alpha) \frac{l^2}{2} \frac{\partial Q}{\partial x} \right) d\sigma = \text{const.}$$

Thus if a value of  $\alpha$  can be found for which (3.9.15a) is satisfied, and if it is also the case that

$$(\mathcal{V} - \alpha) \frac{\partial Q}{\partial x} \geq 0 \quad (3.9.15b)$$

for each  $y$ , then the two grouped terms in the integrand are non-negative. For an infinitesimal perturbation to the basic flow, the positive constant on the right-hand side is arbitrarily small. The integral of  $e_w - \alpha m_w$ , must then be bounded by an arbitrarily small positive constant, say  $\hat{\varepsilon}^2$ :

$$\begin{aligned} \hat{\varepsilon}^2 &\geq \iint_{\mathcal{A}} \left( \mathcal{D}(u'^2 + v'^2) + 2(\mathcal{V} - \alpha)v'd' + d'^2 \right) d\sigma \\ &\geq \iint_{\mathcal{A}} \left( (\mathcal{D}^{1/2}v' - d')^2 + \mathcal{D}u'^2 \right) d\sigma \end{aligned} \quad (3.9.16)$$

in view of the provision (3.9.15a). The transverse velocity  $u'$  must therefore be arbitrarily small, which rules out *shear* instability; that is, instability associated with the transverse motion of the fluid. An instability involving the growth of only  $v'$  and  $d'$  is still possible, but this would require  $d' = \mathcal{D}^{1/2}v'$ . This possibility can be eliminated by an argument explored in Exercise 4.

The two provisions in (3.9.15) therefore comprise a sufficient condition for stability: Ripa's Theorem. The first provision relates to gravity wave propagation while the second, which is identical to Fjortoft's condition for stability, relates to potential vorticity wave propagation.

#### *d. Rotating Channel Flow with Uniform Potential Vorticity*

For the Whitehead et al. (1974) and Gill (1977) models, and other models of rotating channel flow with constant potential vorticity, the second requirement (3.9.15b) of Ripa's sufficient condition for stability is satisfied. The first requirement (3.9.15a) is essentially that a frame of reference  $dy/dt = \alpha$  can be found such that all Froude numbers become less than one. A graphical interpretation of this condition can be obtained by plotting the profiles of  $\pm \mathcal{D}^{1/2}$  and  $\mathcal{V}$ . The requirement is satisfied if one can shift the  $\mathcal{V}$  profile up or down so that it fits between the curves for  $\pm \mathcal{D}^{1/2}$  (Figure 3.9.3a). There is a range of states with uniform potential vorticity, in channels with rectangular cross sections, that satisfy this condition. However, this range has not been mapped out and it is not clear whether connections with the hydraulic and stability properties of the flow exist.

If the depth goes to zero at one or both edges of the channel (Figure 3.9.3b) then the condition is nearly impossible to satisfy. The value of  $\alpha$  must be chosen as the velocity at the edge where the depth vanishes. Then if the depth vanishes at both edges, and the edge velocities differ, the condition cannot be satisfied. Thus, the majority of flows in the Borenas and Lundberg (1986) theory for a parabolic cross-section, and models with other rounded cross-sections, generally do not satisfy the theorem and may be unstable.

*e. Modal disturbances.*

Let

$$\begin{pmatrix} u' \\ v' \\ d' \end{pmatrix} = \text{Re} \left[ \begin{pmatrix} \hat{u}(x) \\ \hat{v}(x) \\ \hat{d}(x) \end{pmatrix} e^{il(y-ct)} \right] + O(\varepsilon),$$

Substitution into (3.9.7a-c) then leads to

$$[il(\mathcal{V} - c)\hat{u}] - \hat{v} = -\frac{d}{dx}\hat{d} \quad (3.9.17a)$$

$$il(\mathcal{V} - c)\hat{v} + \left(1 + \frac{\partial \mathcal{V}}{\partial x}\right)\hat{u} = -il\hat{d} \quad (3.9.17b)$$

and

$$il(\mathcal{V} - c)\hat{d} + \frac{\partial}{\partial x}(\mathcal{D}\hat{u}) + il\mathcal{D}\hat{v} = 0. \quad (3.9.17c)$$

Elimination of  $\hat{u}$  and  $\hat{v}$  in favor of  $\hat{d}$  leads to

$$\frac{d}{dx} \left( \frac{\mathcal{D}}{R} \frac{d}{dx} \hat{d} \right) + \left[ \frac{1}{\mathcal{V} - c} \frac{d}{dx} \left( \frac{\mathcal{D}}{R} \right) - l^2 \frac{\mathcal{D}}{R} - 1 \right] \hat{d} = 0 \quad (3.9.18)$$

where

$$R = 1 + \frac{\partial \mathcal{V}}{\partial x} - l^2 (\mathcal{V} - c)^2$$

The boundary conditions are

$$\mathcal{D}\hat{u} = 0 \quad (\text{edges of flow}). \quad (3.9.19)$$

There are apparently no formal results informing solutions to the eigenvalue problem (3.9.18 and 3.9.19). However, numerical solutions in the long-wave limit generally reveal the presence of two Kelvin-like edge waves and an indeterminate number of

potential vorticity waves. The latter are eliminated when the potential vorticity is uniform. The solutions presented in Figures 2.11.13 and 2.11.14 for the Faroe-Bank Channel are one example, although these were computed using a slightly different formulation. The phase speeds of the potential vorticity waves in this case are bounded above and below by the Kelvin waves speeds. Some of the potential vorticity waves are unstable. At finite wave lengths, a group of inertia-gravity (or Poincaré) waves is present as well. An example of the latter will be discussed below.

The analysis is particularly simplified in the case of zero potential vorticity ( $Q=0$ ). Equation (3.9.17b) reduces to

$$(\mathcal{V} - c)\hat{v} = -\hat{d}. \quad (3.9.20)$$

Also, the perturbation potential vorticity  $q' = \mathcal{D}^{-1}(d\hat{v}/dx - il\hat{u} - Q\hat{q})$  must vanish:

$$\frac{d\hat{v}}{dx} = il\hat{u} \quad (3.9.21)$$

If these last two relations are used eliminate  $\hat{v}$  and  $\hat{d}$  from (3.9.17c), one finds

$$\frac{d}{dx} \left( \mathcal{D} \frac{d\hat{v}}{dx} \right) - l^2 [\mathcal{D} - (\mathcal{V} - c)^2] \hat{v} = 0 \quad (3.9.22)$$

In view of (3.9.21) the boundary condition  $\mathcal{D}\hat{u}=0$  implies that  $\mathcal{D}d\hat{v}/dx = 0$  at the edges. Integration of (3.9.22) across the flow then yields

$$l^2 \int [\mathcal{D} - (\mathcal{V} - c)^2] \hat{v} dx = 0, \quad (3.9.23)$$

where the integrations is understood to be across the width of the basic flow, whether or not vertical sidewalls are present.

Now let  $c=c_r+ic_i$ , so that  $c_i>0$  implies instability. The values of  $c_r$  and  $c_i$  can be bounded according to a ‘semicircle’ theorem, first derived by Howard (1961) in connection with stratified shear flow and extended by Hayashi and Young (1987) to an equatorial, shallow-water flow. Multiply (3.9.22) by the complex conjugate  $\hat{v}^*$  of  $\hat{v}$ , integrate the result across the channel, and apply the boundary conditions to obtain

$$\int \left\{ [\mathcal{D} - (\mathcal{V} - c)^2] |\hat{v}|^2 + l^{-2} \mathcal{D} |d\hat{v}/dx|^2 \right\} dx = 0.$$

The real and imaginary parts of this relation are

$$\int \left\{ \left[ \mathcal{D} - (\mathcal{V} - c_r)^2 + c_i^2 \right] |\hat{v}|^2 + l^{-2} \mathcal{D} |d\hat{v} / dx|^2 \right\} dx = 0 \quad (3.9.24)$$

$$l^2 c_i \int (\mathcal{V} - c_r) |\hat{v}|^2 dx = 0 \quad (3.9.25)$$

Now let  $\mathcal{V}_{\min} \leq \mathcal{V} \leq \mathcal{V}_{\max}$  and suppose that  $c_i > 0$ . Then a series of inequalities (Exercise 5) leads to

$$\int \left\{ \left[ c_r + \frac{1}{2} (\mathcal{V}_{\max} + \mathcal{V}_{\min}) \right]^2 + c_i^2 - \left[ \frac{1}{2} (\mathcal{V}_{\max} - \mathcal{V}_{\min}) \right]^2 \right\} |\hat{v}|^2 dx + \int \mathcal{D} |\hat{v}|^2 + l^{-2} \int \mathcal{D} |d\hat{v} / dx|^2 dx \geq 0 \quad (3.9.26)$$

The second and third integrals are non-negative and instability therefore requires

$$\left[ c_r + \frac{1}{2} (\mathcal{V}_{\max} + \mathcal{V}_{\min}) \right]^2 + c_i^2 \leq \left[ \frac{1}{2} (\mathcal{V}_{\max} - \mathcal{V}_{\min}) \right]^2. \quad (3.9.27)$$

The complex phase speed of an unstable wave must therefore fall within the semi-circle shown in Figure 3.9.4.

Of particular interest in hydraulics is the stability long waves. Let  $l \ll 1$  and write

$$\hat{v} = v_0 + lv_1 + l^2 v_2 + \dots \quad (3.9.28)$$

For simplicity, we will normalize  $\hat{v}$  such that its maximum value is unity.

We will now restrict attention to a current that vanishes at the two edges. Then the lowest order approximations to (3.9.22) and (3.9.23) are

$$\frac{d}{dx} \left( \mathcal{D} \frac{dv^{(0)}}{dx} \right) = 0$$

and

$$\int [\mathcal{D} - (\mathcal{V} - c_o)^2] v_0 = 0.$$

Integration of the first relation and enforcement of the boundary conditions leads to

$$v^{(0)} = \text{const.} = 1.$$

and the second relation then yields

$$c_0^2 - 2c_0 \langle \mathcal{V} \rangle + \langle \mathcal{V}^2 \rangle - \langle \mathcal{D} \rangle = 0.$$

The brackets denote a cross-channel average. The phase speeds of the two waves are given by

$$c_o = \langle \mathcal{V} \rangle \pm \left[ \langle \mathcal{V} \rangle^2 - \langle \mathcal{V}^2 \rangle + \langle \mathcal{D} \rangle \right]^{1/2}. \quad (3.9.29)$$

Long wave instability occurs for  $\langle \mathcal{V} \rangle^2 - \langle \mathcal{V}^2 \rangle + \langle \mathcal{D} \rangle < 0$ .

For real  $c_o$ , (3.9.29) suggests the Froude number

$$F_o = \frac{\langle \mathcal{V} \rangle}{\left[ \langle \mathcal{V} \rangle^2 - \langle \mathcal{V}^2 \rangle + \langle \mathcal{D} \rangle \right]^{1/2}} \quad (3.9.30).$$

The flow is hydraulically critical when  $F_o=1$ . These results hold for general bottom topography.

*e. The GKS instability.*

A example of a instability that acts in the presence of uniform potential vorticity, and therefore does not draw energy from the mean, was analyzed by Griffiths, Killworth and Stern (1982). As shown in Figure 3.9.5a, the basic flow rides over a constant bottom slope  $dx/dx=S$  and has zero potential vorticity. [Paldor (1983) treated the special case  $S=0$ .] The basic flow profile is computed from the geostrophic relation and from the zero-potential vorticity constraint  $\partial \mathcal{V} / \partial x = -1$ . If basic current is positioned so that  $x=0$  lies midway between the two edges, and if the scale depth  $D$  is chosen as the centerline depth, then the basic velocity and layer thickness are given by

$$\mathcal{V} = S - x,$$

$$\mathcal{D} = 1 - \frac{x^2}{2},$$

The edges of the current therefore lie at  $x=\pm\sqrt{2}$ .

The speeds of the two long waves of the flow can be calculated from (3.9.29) using  $\langle \mathcal{V} \rangle = S$ ,  $\langle \mathcal{V}^2 \rangle = S^2 + 2/3$ , and  $\langle \mathcal{D} \rangle = 2/3$ . Both waves have the same speed:

$$c_o = S.$$

or  $c_o^*=Sg'/f$ .

If one attempts to calculate the next term in the wave number expansion (3.9.28) the eigenfunction is again found to be a constant. Our normalization requires this constant to be zero. It can then be shown that the integral determining the first correction  $c_1$  to the wave speed is degenerate, and thus one must go to the next order of approximation. At  $O(l^2)$  (3.9.22) and (3.9.23) give

$$\frac{d}{dx} \left( \mathcal{D} \frac{dv_2}{dx} \right) = - \left[ \mathcal{D} - (\mathcal{V} - c_0)^2 \right]$$

and

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left\{ \left[ \mathcal{D} - (\mathcal{V} - c_0)^2 \right] v_2 + [2(\mathcal{V} - c_0)c_2 - c_1^2] v_o \right\} dx = 0.$$

Substituting the solution to the first relation into the second leads, after a bit of algebra, to

$$c_1 = \pm \frac{2i}{\sqrt{15}}.$$

Waves with long, but finite, lengths are therefore unstable.

For the growing wave ( $c_1 = +2i / \sqrt{15}$ ), it can also be shown that positions of the right and left edges of the current (at  $t=0$ , say) are given by

$$\begin{pmatrix} b(y,t) \\ -a(y,t) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix} + \varepsilon \begin{pmatrix} \cos(y) - \frac{2l}{\sqrt{30}} \sin(y) \\ \cos(y) + \frac{2l}{\sqrt{30}} \sin(y) \end{pmatrix},$$

where  $\varepsilon$  is again a measure of the wave amplitude. Thus the original long wave ( $l=0$ ) has a meandering structure: it experiences displacements that are equal and in phase on either side of the flow. The lowest order correction introduces excursions that are equal but out of phase. This structure can be seen to some extent in the early stages of the instability as captured in a laboratory experiment (Figure 3.9.6).

Numerical solutions of the eigenvalue problem show that the central ingredients of the long wave instability are preserved well into the range of finite  $l$ . As shown in Figure 3.9.7, the unstable wave continues to have  $c_r=S$  and the growth rate  $lc_i$  increases with increasing  $l$ , reaching a maximum value of about .15 around  $l=0.8$ . The most unstable wave therefore has a wavelength of about 8 deformation radii and will double in amplitude over several rotation periods. Both features are characteristic of the laboratory experiment (Figure 3.9.6), where the initial current width is about 3.5 deformation radii, the wave length is roughly twice that, and the instability reaches a large amplitude in 8 rotation periods. Although the instability disappears when  $l$  exceeds a value  $l_c \approx 1.1$ , Hayashi and Young (1987) have shown that isolated bands of instability (the small lobes

in Figure 3.9.7b) with smaller growth rates reappear at larger  $l$ . These weaker instabilities are shown as small lobes along the  $l$  axis.

The growth mechanism for the GKS instability is clarified by consideration of the phase speed curves shown Figure 3.9.7a. For  $l$  slightly greater than the cutoff value  $l_c$ , there are two neutral waves with phase speeds slightly greater and less than  $S$ . Analysis of the horizontal structure of these two shows that they are closely related to Kelvin waves: the faster is trapped to the right edge and the slower to the left edge of the flow. Where  $l=l_c$  the values of  $c_r$  merge and the two wave resonate. Other bands of instability are similarly interpreted—they arise when the phase speeds of two neutral waves merge. One of the waves is generally of the Kelvin type and the other a modified inertia-gravity type (corresponding to the remaining curves in Figure 3.9.7a).

Direct calculation of the disturbance energy  $E_d$  (also the wave energy for this case) for the waves shows that one member of a merging pair has negative and the other positive energy. In fact, it can be shown that the energy is opposite in sign to  $c_r^{-1}dc_r/dl$  and thus the two members of any pair must have opposite signed  $E_d$ . For the unstable disturbance produced by the interaction between the two members, the disturbance energy is zero by definition. The potential vorticity gradient is zero for this flow and thus the mean energy  $E_m$  associated with the disturbance is also zero. The unstable pair does not draw on energy from the mean; instead, growth in the positive  $E_d$  of one member is offset by growth in the negative energy of the other. A similar result holds for the disturbance momentum.

GKS have shown that the long wave instability acts when the potential vorticity of the background flow is arbitrary. They compute the growth rates for several cases of uniform (non-zero) potential vorticity  $f/D_\infty$ . The background flow for this last case (Figure 3.9.5b) is similar to that of the Gill (1977) model. In dimensionless terms, there is a central region with uniform depth  $D_\infty$ , now moving at speed  $g'S/f$ , and flanked by boundary layers of dimensional thickness  $(gD_\infty)^{1/2}/f$ . When the width  $W$  of the whole current is wide compared to the latter, the modified Kelvin waves are trapped to the edges of the flow and the coupling is weak, as is the instability. When the width and deformation radius are comparable, the coupling is strong and, the system behaves more or less as the in the zero potential vorticity limit. Readers familiar with the classical Eady (1949) model of baroclinic instability will see similarities with the present problem. Both models involve edge waves that are separated by an interior region. (In the Eady problem the 'edges' are rigid, horizontal, upper and lower boundaries.) Were it not for a background flow, the waves would propagate in opposite directions and would not couple. The tendency of the waves is to propagate in opposite directions, but the sheared background flow can, over a certain range, bring the two speeds into equality. The waves then couple and experience resonant growth. The effect weakens as the upper and lower boundaries are separated.

*f. Connections with hydraulic theory.*



The history of hydraulics, particularly with respect to flow criticality, is replete with tantalizing but vague connections to instability theory. However, it has proven difficult to make definitive statements about such connections. As an example, consider the Fj rtoft sufficient condition for stability (also the second requirement (3.9.15b) of Ripa's condition for stability). It states that a necessary condition for *instability* of a single layer, quasigeostrophic flow is that  $(V - \alpha)(\partial Q / \partial x) < 0$ . Thus the potential vorticity must increase in the direction to the left of the velocity seen in the moving frame. High potential vorticity on the left suggests that potential vorticity waves attempt to propagate counter the background flow, at least in simplified models. The rest frame ( $\alpha=0$ ) version of this condition also the requirement in the Pratt and Armi (1988) model for flow criticality with respect to potential vorticity waves (see Section 2.9). The first requirement (3.9.15a) of Ripa's theorem also appears to intersect with hydraulic theory in requiring the flow to be subcritical in a moving reference frame. Just how strong these connections are is not known.

Another connection between flow instability and hydraulic criticality is suggested by the physical mechanism of the GKS instability. Consider a steady flow that is evolving gradually in the  $y$ -directions and that becomes unstable to long waves downstream of some location  $y_0$ . If the instability results from the resonant coupling of two neutral long waves, then the corresponding wave speeds  $c_1$  and  $c_2$  must equal each other at  $y_0$ . The flow there must then be supercritical with respect to these waves, at least in the sense that information carried by the waves moves in one direction. It is also possible, though less likely, that the flow is critical, with  $c_1=c_2=0$ . In any case, the flow cannot be subcritical with respect to the two waves. The importance of this property is the suggestion that long-wave instability may be confined to regions of supercritical flow in a wide range of applications. An example that will be reviewed in detail is two-layer flow in a non-rotating channel (see Section 5.2).

## Exercises

1. *Derivation of the equation for conservation of total momentum.* Begin the flux form (see Section 3.5) of the  $y$ -momentum equation with  $dh/dy=0$ :

$$\frac{\partial}{\partial t}(vd) + \frac{\partial}{\partial y}\left(v^2d + \frac{1}{2}d^2\right) + \frac{\partial}{\partial x}(uvd) + ud = 0 ,$$

obtained by multiplying (2.16) by  $d$  and using (2.1.7). Then write

$$ud = \frac{\partial}{\partial x}(xud) - x \frac{\partial}{\partial x}(ud) ,$$

use (2.1.7) again, and integrate the result of  $\mathcal{A}$  to (3.9.4).

2. Derive the equation (3.9.8) for the wave energy density. One method follows this plan:

(a) Begin by taking  $uD \times (3.9.7a) + vD \times (3.9.7b) + dD \times (3.9.7c)$ , which should give the intermediate equation

$$\frac{\partial e_w}{\partial t} + \mathcal{V} \frac{\partial}{\partial x} (\mathcal{D} \overline{uv}) = - \frac{\partial}{\partial x} (\mathcal{V} \mathcal{D} \overline{uv} + \mathcal{D} \overline{ud})$$

(b) Then write out the definition of the potential vorticity flux  $\overline{vq}$ , rearrange some derivatives, and use the  $x$ -momentum equation to simplify. Substitution of the result for the second term on the left-hand side of the equation in (a) leads to the desired result.

3. Show that the wave energy  $e_w$  is non-negative provided that  $\mathcal{V}^2/\mathcal{D} \leq 1$  for all  $y$ .

4. *Completion of the proof of of Ripa's theorem.* Show that the relationship  $\mathcal{D}^{1/2}v = d$  would prevent satisfaction of the both boundary conditions, whether free edges or vertical wall as present.

5. *On the derivation of the semicircle theorem.* With  $\mathcal{V}_{\min} \leq \mathcal{V} \leq \mathcal{V}_{\max}$  observe that

$$0 \geq \int (\mathcal{V} - \mathcal{V}_{\min})(\mathcal{V} - \mathcal{V}_{\max}) |\hat{v}|^2 dx = \int \mathcal{V}^2 |\hat{v}|^2 dx - (\mathcal{V}_{\min} + \mathcal{V}_{\max}) \int \mathcal{V} |\hat{v}|^2 dx + \mathcal{V}_{\min} \mathcal{V}_{\max} \int \mathcal{V} |\hat{v}|^2 dx$$

Next show using equations ( 2.9.24 ) and ( 2.9.25 ) that

$$\int [\mathcal{D} - \mathcal{V}^2 + c_r^2 + c_i^2] |\hat{v}|^2 dx + l^{-2} \mathcal{D} \int |d\hat{v}.dx|^2 dx = 0 .$$

Using this last relation and (2.9.25) to substitute for the first two, right-hand integrals in the first equation, obtain (2.9.26).

## Figure Captions

Figure 3.9.1 Cross section of the basic flow.

Figure 3.9.2 Periodic and amplifying disturbances of a simple pendulum.

Figure 3.9.3 (a) Graphical representation of one of the two requirements (see 3.9.15a) of Ripa's Theorem. Stability requires that the velocity profile can be shifted up or down to fit entirely in the shaded area.

Figure 3.9.4 The semicircular region of the complex phase speed plane in which a growing wave must lie. (After Howard, 1961)

Figure 3.9.5 (a) The basic flow of the Griffiths, et al. (1982) stability model: a zero potential vorticity current over a sloping bottom. (b) Schematic view of a flow of uniform, non-zero, potential vorticity flow along a constant slope.

Figure 3.9.6 Streak photos showing the instability of a flow set up by introduction of an annular region of buoyant fluid at the upper boundary of a much deeper fluid. The initial width of the flow is approximately 3.4 deformation radii (based on the initial thickness of the buoyant layer). Photos *a-d* were taken at 2,4,6, and 8 revolutions following release of the fluid. (Figure 8 from Griffiths, et al. 1982).

Figure 3.9.7. The phase speed (a) and growth rate (b) of instabilities of a zero potential vorticity current on a sloping bottom (from Hayashi and Young, 1987). The GLK instability corresponds to the band roughly spanning  $0 < l < 1.1$ . Instabilities isolated bands of instability at higher  $l$  are due to unstable resonance between Kelvin-like and inertia-gravity waves. The  $\sqrt{2}$  is due to a discrepancy between the present scaling and that of Hayashi and Young.

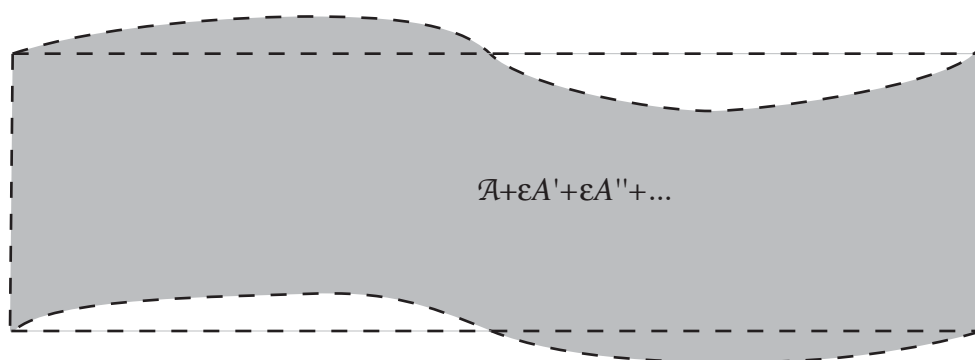
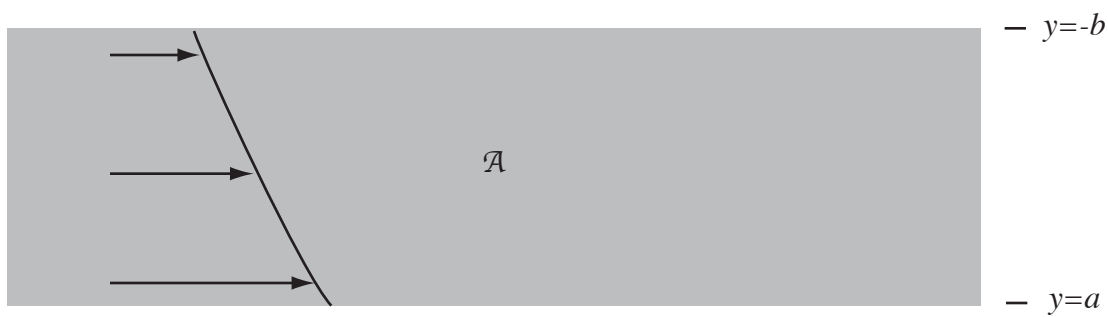
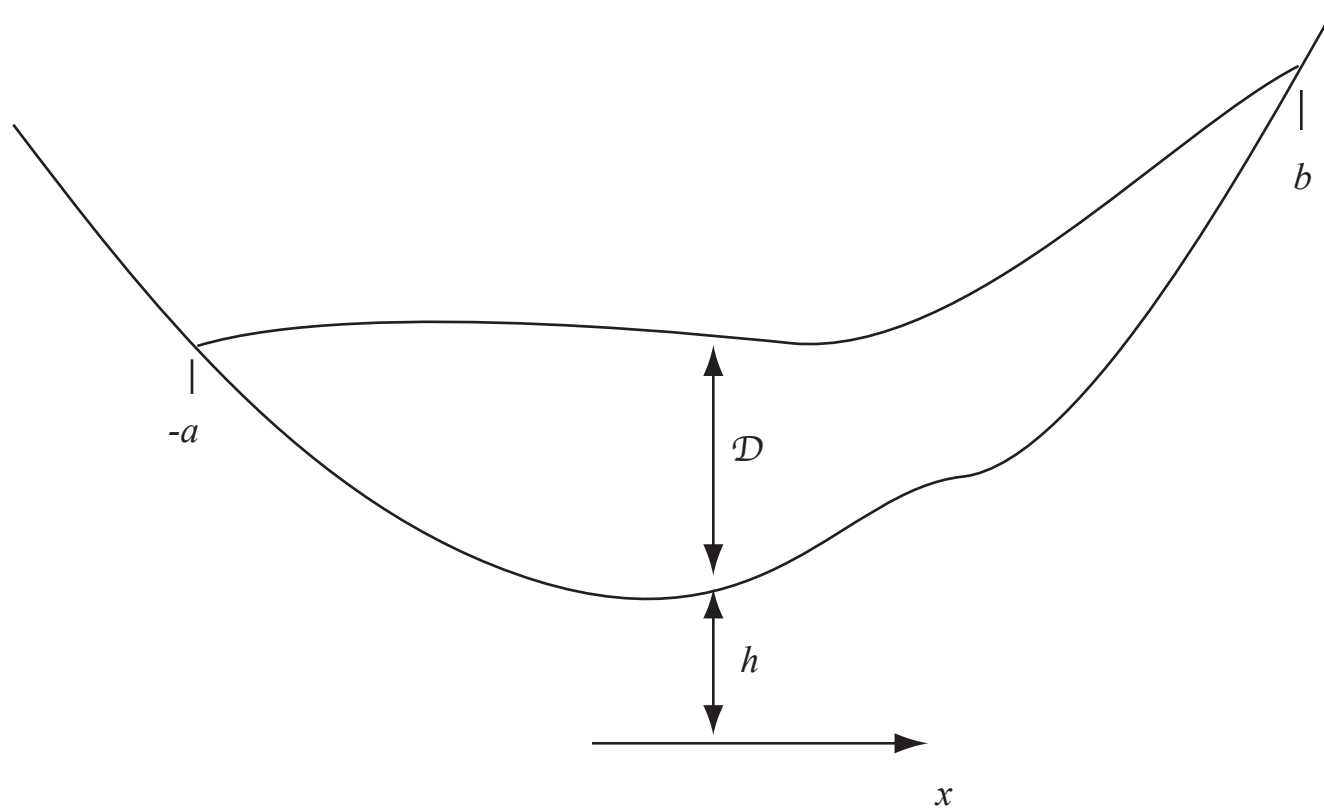


Figure 3.9.1

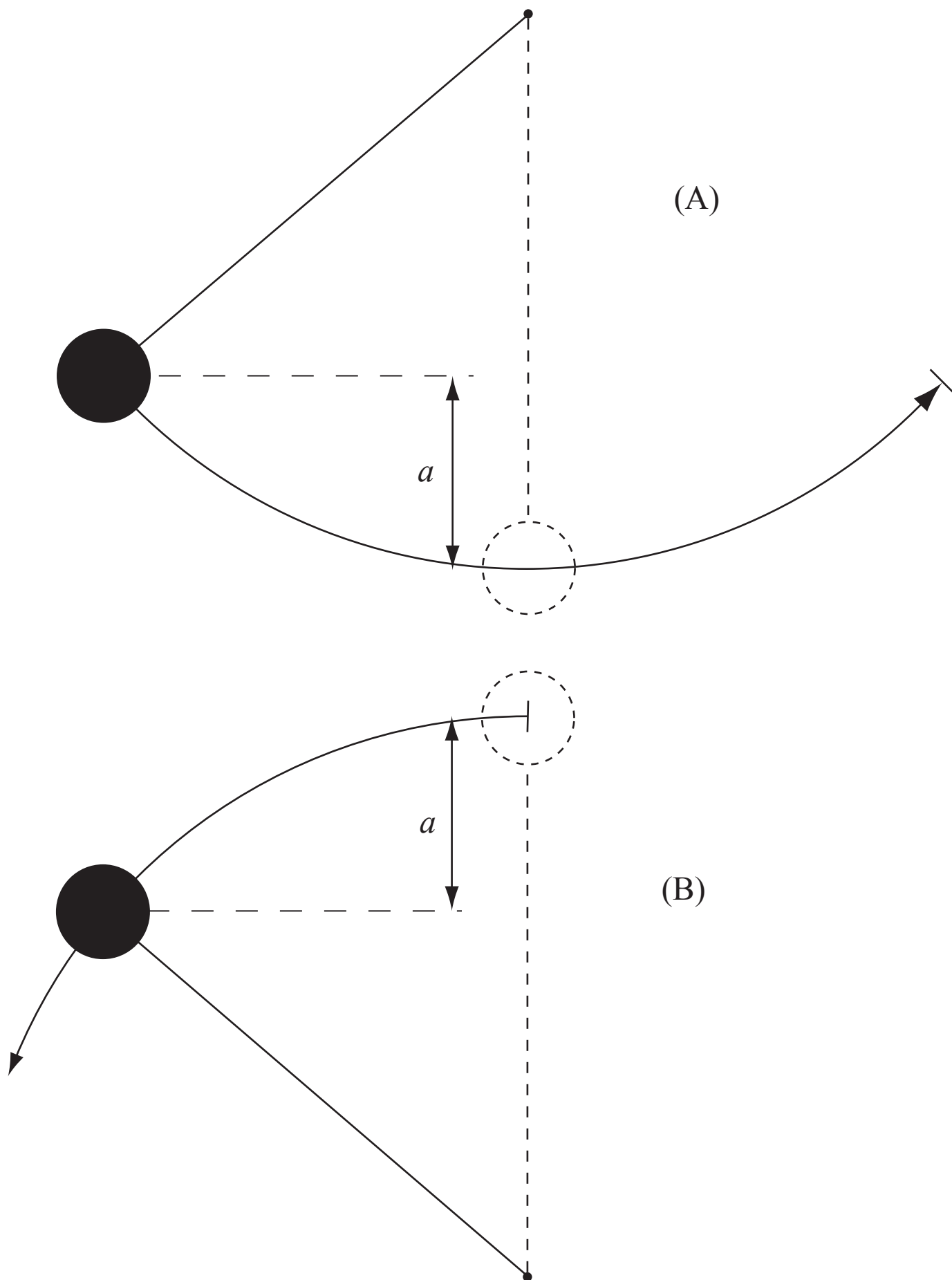


Figure 3.9.2

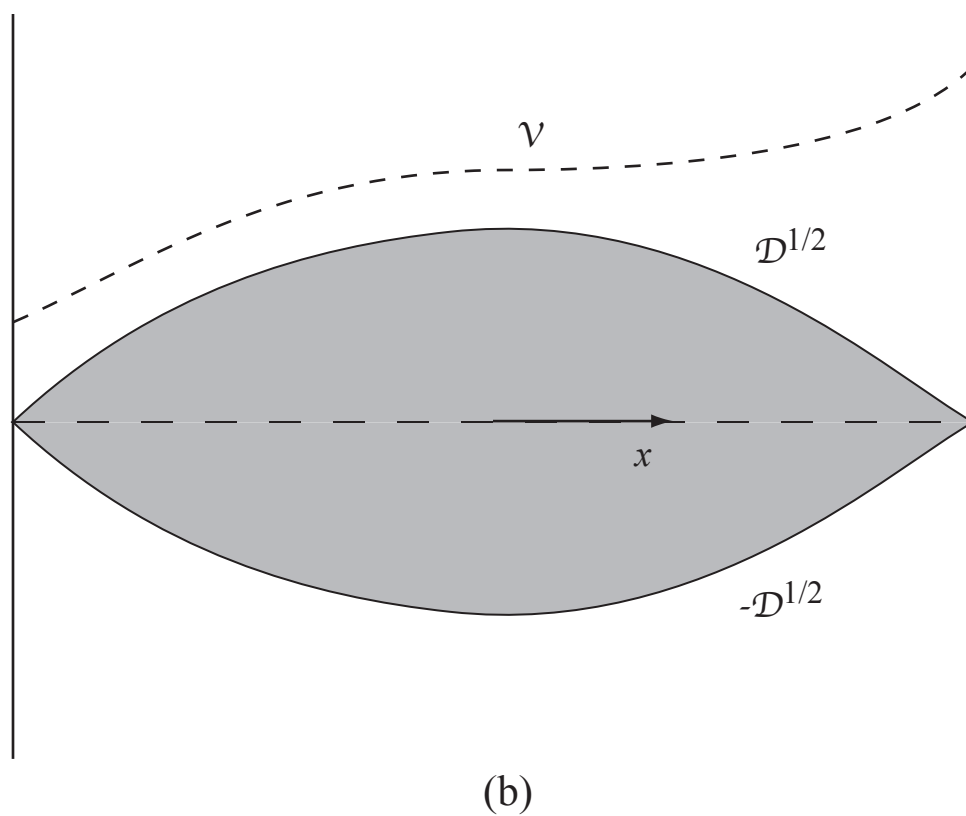
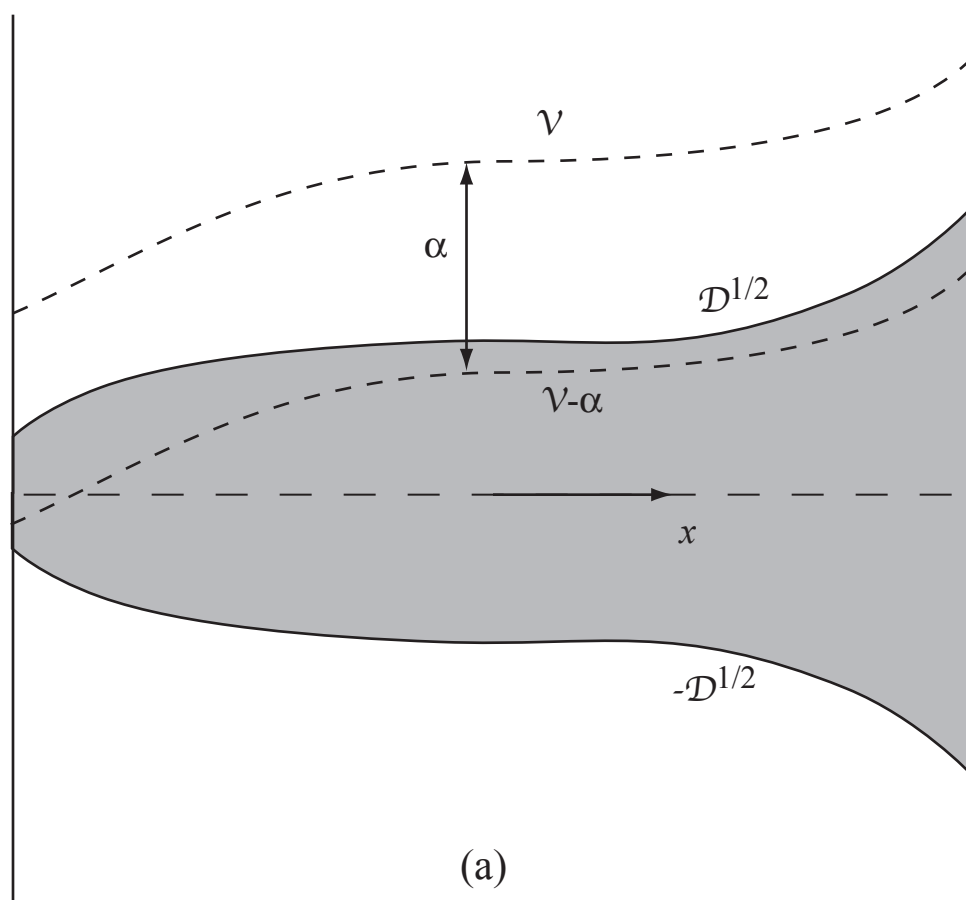


Figure 3.9.3

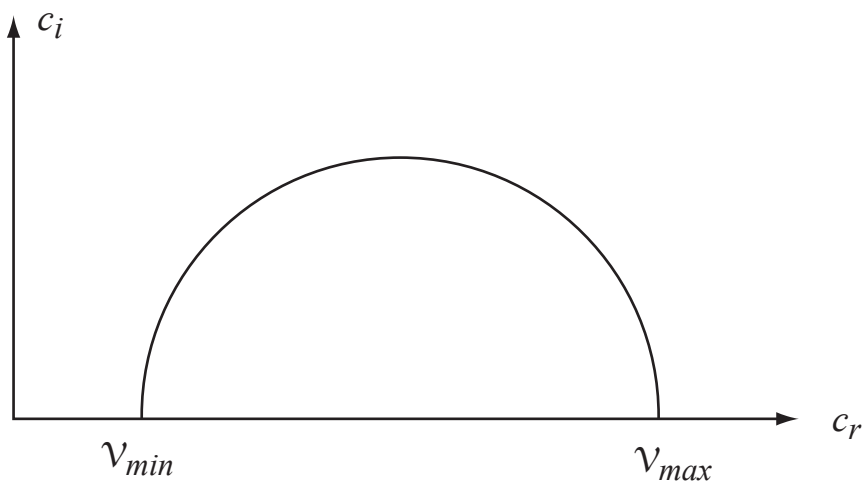
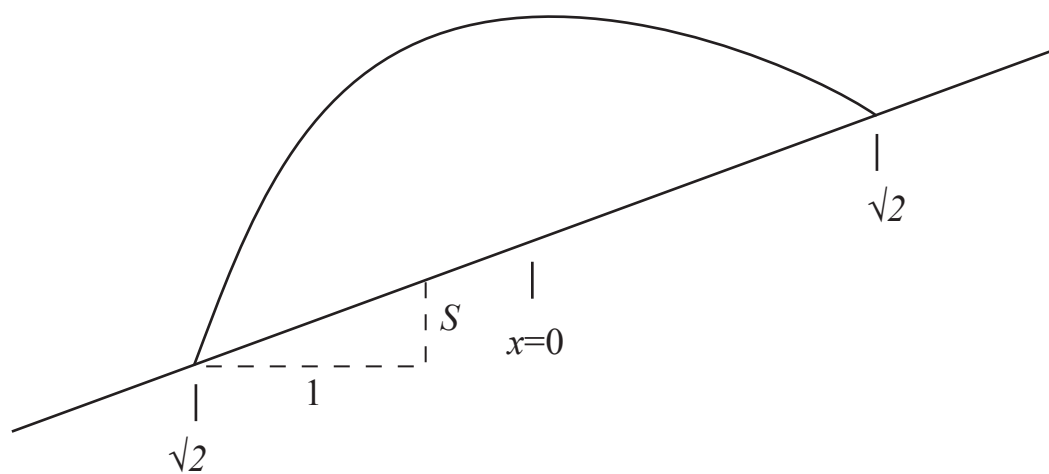
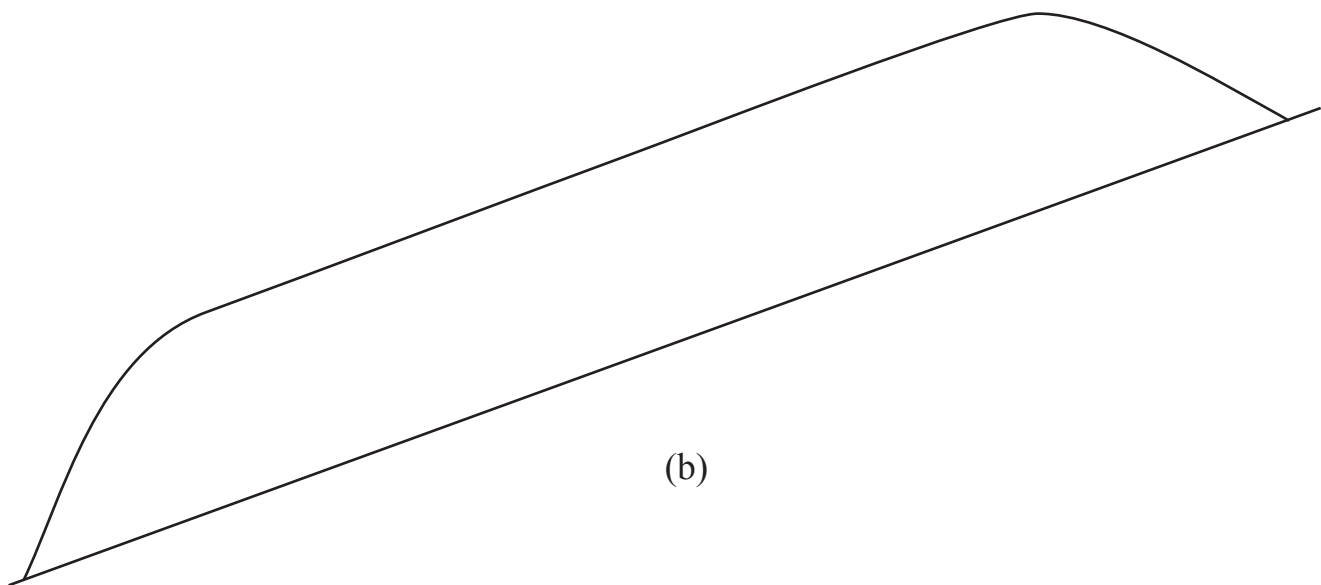


Figure 3.9.4



(a)



(b)

Figure 3.9.5



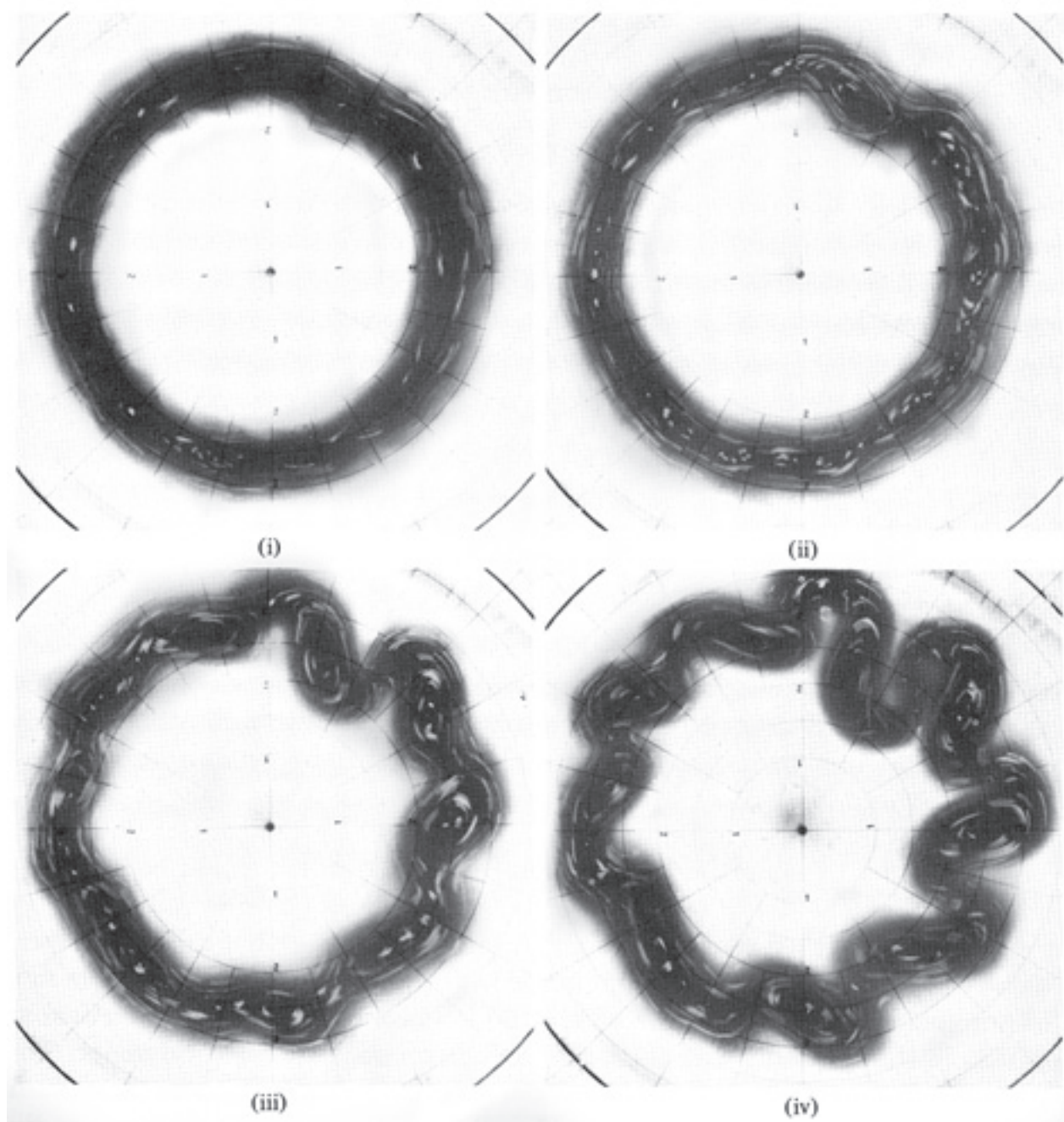


Figure 3.9.6 (low resolution version)

