5.2 Basic Theory for a Nonrotating Channel.

There are several articles that deserve special mention in the annals of two-layer hydraulics, the earliest being Stommel and Farmer's (1952, 1953) model of estuary dynamics. Many of the distinctive properties of these flows, including the possibility of two control sections, were identified by Wood (1968, 1970) in his laboratory simulations of lock exchange between basins and selective withdrawal from stratified reservoirs. The steady theory was unified and extended in a series of articles by L. Armi and D. Farmer, including Armi (1986), Armi and Farmer (1986, 1987, 1988) and Farmer and Armi (1986), who were interested in the Strait of Gibraltar and other oceanographic examples of exchange flow. This work forms the foundation for our summary and their fingerprints are on much of what follows. A slightly different view is provided by Long's (1954) towing experiments and subsequent investigations of initial-value problems by various authors (Baines, 1995 and references contained therein). This literature gives considerable insight into how two-layer flows are set up.

The governing equations are the x*-independent, f=0 versions of (5.1.4,5.1.6, and 5.1.7). These equations can be put into characteristic form [Baines (1995) pp. 98-99] using the methods laid out in Appendix B. The characteristic speeds are given by

$$c_\pm{}^* = \frac{v_1{}^* d_2{}^* + v_2{}^* d_1{}^*}{d_1{}^* + d_2{}^*} \pm \left\{ \frac{g' d_1{}^* d_2{}^*}{d_1{}^* + d_2{}^*}\left[1 - R_b^{-1}\right] \right\}^{1/2} . \qquad (5.2.1)$$

For an evolving flow containing disturbances of arbitrary amplitude, we may regard c_+* or c_-* as the local and instantaneous speed of a signal propagating forward or backward with respect to the advective speed defined by the first expression on the right-hand side. Although no linearization has been made, we can also regard c_+* and c_-* as the speeds of small-amplitude, long waves propagating on a steady and uniform background flow with depth and velocity d_i* and v_i*. Note that these speeds are real only so long as

$$R_b = \frac{g'(d_1{}^* + d_2{}^*)}{(v_1{}^* - v_2{}^*)^2} \geq 1 . \qquad (5.2.2)$$

Thus, if the magnitude of the *shear velocity* $v_1{}^* - v_2{}^*$ is large enough, $c_\pm{}^*$ become imaginary, corresponding to long-wave Kelvin-Helmholtz instability of the background flow. The parameter R_b is a discrete (or 'bulk') form of the *Richardson number* $R_i = [g\rho^{-1}\partial\rho / \partial z^*] / (\partial v^* / \partial z^*)^2$ for continuously stratified shear flow.

The possibility of instability is an important departure from the behavior of the single-layer case considered in the first chapter. It is natural to ask whether traditional properties such as hydraulic control and upstream influence remain meaningful when part or all of the flow is unstable. The answer to this question is largely unknown at the time of this writing. For many of the two-layer flows encountered in nature or in the laboratory, the primary instabilities occur in supercritical regions away from control sections. The associated disturbances propagate away from the control section(s) and conditions there remain steady.

There is another aspect of the stability issue that bears consideration. An analysis (e.g. Turner 1973, Sec. 4.1) of the inviscid, two-layer system with respect to an arbitrarily short (nonhydrostatic) disturbances shows that the flow is *always* unstable provided that $v_1^* \neq v_2^*$. In a two-layer system with infinite layer depths, for example, all sinusoidal interfacial waves with lengths less than $\pi|v_1^*-v_2^*|/g'$ are unstable. The resulting mixing can destroy the sharp interface and create an intermediate transitional layer. Wilkinson and Wood (1985) present a laboratory demonstration using a hydraulically driven, two-layer system. If the shear is weak, unstable waves have small scales and the intermediate layer remains thin. Its thickness d_I^* can be estimated using the hypothesis that the layer will grow until the mean flow becomes stable. A necessary condition for instability of a thin, laminar, intermediate layer is that the bulk Richardson number $g'd_I^*/(v_1^*-v_2^*)^2$ based on d_I^* falls beneath 1/4. Empirical evidence (e.g. Thorpe 1973 and Koop and Browand 1979) suggests a transitional value closer to 0.3, and thus the expected layer thickness is

$$d_I^* \cong 0.3(v_1^*-v_2^*)^2/g'.$$

As long as d_I^* remains much less than $d_1^*$and d_2^* the presence of the intermediate layer may to a first approximation be disregarded and the two-layer protocol adopted.

Some of the important differences between single- and two-layer hydraulics may be anticipated from an examination of the formula for the long-wave phase speed. If the background flow is at rest, (5.2.1) reduces to

$$c_\pm^* = \pm\left(\frac{g'd_1^*d_2^*}{d_1^*+d_2^*}\right)^{\frac{1}{2}} . \qquad (5.2.3)$$

When the lower layer is relatively thin ($d_2^* << d_1^*$), $c_\pm^*$ reduces to the value $\pm\sqrt{g'd_2^*}$ for a single layer under reduced gravity. A corresponding result for the upper layer is obtained by taking $d_2^* << d_1^*$. If the total depth $d_2^*+d_1^*$ is held constant while the interface is varied from the top to bottom boundary, $|c_\pm^*|$ vary from zero to their maximum values at mid-depth ($d_1^* = d_2^*$), then back to zero. This is quite different from the case of a resting single layer, in which $|c_\pm^*|$ increases monotonically as the lower layer depth increases.

From (5.2.1) it can be shown that

$$c_+^*c_-^* = \frac{g'd_1^*d_2^*}{d_1^*+d_2^*}\left(\frac{v_1^{*2}}{g'd_1^*}+\frac{v_2^{*2}}{g'd_2^*}-1\right), \qquad (5.2.4)$$

and thus at least one of the characteristic speeds is zero if the sum of the layer Froude numbers,

$$F_1 = \frac{v_1*}{(g'd_1*)^{1/2}} \text{ and } F_2 = \frac{v_2*}{(g'd_2*)^{1/2}}, \tag{5.2.5}$$

is unity. This result makes it convenient to define a *composite* Froude number G such that

$$G^2 = F_1^2 + F_2^2, \tag{5.2.6}$$

Critical flow corresponds to G^2=1, implying that one or both of $c_\pm*$ is zero. If G^2<1 then (5.2.4) indicates that the product of c_+* and c_-* is <0, implying that the two internal gravity waves propagate in opposite directions. This type of flow is considered *subcritical* since information can move in both directions. Similarly, G^2>1 implies that both waves propagate in the same direction and the flow is *supercritical*. These definitions avoid reference to 'upstream' or 'downstream', a tacit acknowledgement that two layers may flow in opposite directions. Thus, supercritical flow may have both waves moving in the +$y*$ direction or *vice versa*. It is not meaningful to talk about the criticality of an individual layer unless the other layer is inactive. For example, it is not meaningful to state that layer 1 is 'critical' when F_1=1, unless F_2<<1. (However, it can be stated with certainty that the two layer flow is supercritical if either F_1 or F_2 is >1.)

Imagine a flow that is evolving in the $y*$-direction due to changes in the channel geometry and suppose that this flow undergoes a transition from stable to unstable at a particular $y*$. Since R_b=1 at that section (5.2.1) requires that $c_+*=c_-*$ there. Thus the flow must first be critical or supercritical before it can become unstable with respect to long waves. This is a special case of the connection, discussed at the end of Section 3.9, between long-wave instability and critical/supercritical flow.

The volume transport within a layer is

$$Q_i* = v_i*d_i*w* \tag{5.2.7}$$

and both Q_1* and Q_2* are constants for steady flow. If Q_1* and Q_2* have opposite signs we have an *exchange flow*. *Pure exchange flow* occurs when the net or *barotropic* transport

$$Q* = Q_1*+Q_2* \tag{5.2.8}$$

is zero. Another quantity that will prove useful is the transport ratio:

$$Q_r = \frac{Q_1*}{Q_2*}. \tag{5.2.9}$$

The time-dependent continuity equation for a particular layer, which may be obtained by integrating (5.1.7) across the channel, is

$$w\frac{\partial d_i*}{\partial t*}+\frac{\partial Q_i*}{\partial y*}=0\,.$$

An important constraint on the barotropic transport can be formulated by adding together the time-dependent continuity equations for each layer. Noting that d_1*+d_2* depends only on *y:*

$$\frac{\partial(d_1*+d_2*)}{\partial t*}=w*^{-1}\frac{\partial Q*}{\partial y*}=0\,.$$

The total transport *Q** *is* therefore a function of *t** only. It follows that *Q* is constant in time if this is so at any section.

Steady solutions are normally calculated using the internal Bernoulli equation (5.1.17). In thinking about the various solutions, it often helps to imagine that the channel is connected to an infinitely wide basin where the layer depths $d_{1\infty}*$ and $d_{2\infty}*$ are non-zero and where the flow is therefore quiescent. If *h**=0 in this basin then

$$\Delta B*=g'd_{2\infty}*\;. \qquad (5.2.10)$$

If a hydraulic jump occurs within the channel, the value of ΔB* will generally change across the jump.

At this stage, the mathematical problem for the steady two-layer flow involves four variables (the depth and velocity in each layer) governed by two continuity equations (5.2.7), the internal Bernoulli equation (5.1.17), and the geometric constraint

$$d_1*(y*)+d_2*(y*)+h*(y*)=z_T* \qquad (5.2.11)$$

resulting from the rigid-lid assumption. It is possible to reduce the algebra to a single equation for one of the layer thicknesses and sketch solution curves analogous to that shown in Figure 1.4. Another approach is to reduce the algebra to two equations in two variables and sketch solution curves in the two-dimensional space of these variables. The choice of method is largely one of personal preference. Our preference is for the second approach, as developed by Armi (1986) using the layer Froude numbers as the dependent variables. Following his formulation, the layer depths and velocities may be written in terms of F_1 and F_2 using

$$d_i*=\frac{Q_i*^{2/3}}{g'^{1/3}F_i^{2/3}w*^{2/3}} \quad \text{and} \quad v_i*=\left(\frac{Q_i*g'}{w*}\right)^{1/3}F_i^{2/3}. \qquad (5.2.12a,b)$$

Making these substitutions and using (5.2.11) allows (5.1.17) to be written in the form

$$\mathcal{G}_1(F_1,F_2;h^*,w^*)=\left(\frac{g'Q_1{}^*}{w^*}\right)^{2/3}(\tfrac{1}{2}F_1^{4/3}-\tfrac{1}{2}Q_r^{-2/3}F_2^{4/3}+F_1^{-2/3})+\Delta B^*-g'z_T{}^*=0\,. \qquad (5.2.13)$$

Furthermore, (5.2.11) itself can be rewritten as

$$\mathcal{G}_2(F_1,F_2;h^*,w^*)=Q_r^{2/3}F_1^{-2/3}+F_2^{-2/3}-(z_T{}^*-h^*)g'^{1/3}w^{*2/3}\,Q_2{}^{*-2/3}=0\,. \qquad (5.2.14).$$

Using the two-variable generalization of Gill's approach, the critical condition may be calculated using (1.5.9), which leads to

$$\frac{\partial\mathcal{G}_1}{\partial F_1}\frac{\partial\mathcal{G}_2}{\partial F_2}-\frac{\partial\mathcal{G}_1}{\partial F_2}\frac{\partial\mathcal{G}_2}{\partial F_1}=0\,. \qquad (5.2.15)$$

The reader may wish to verify that application to (5.2.13) and (5.2.14) yields the result $G^2=1$, the condition for stationary disturbances derived from the wave speed formula.

The regularity condition that must hold at a critical section can be obtained by applying (1.5.11), which leads to

$$\frac{\partial\mathcal{G}_1}{\partial\gamma_i}\left(\frac{\partial\mathcal{G}_2}{\partial y}\right)_{\gamma_1,\gamma_2}-\frac{\partial\mathcal{G}_2}{\partial\gamma_i}\left(\frac{\partial\mathcal{G}_1}{\partial y}\right)_{\gamma_1,\gamma_2}=0\quad(i{=}1\ or\ i{=}2)\,, \qquad (5.2.16)$$

with the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ defined by (5.2.13) and (5.2.14) and $\gamma_1=F_1{}^{2/3}$ and $\gamma_2=F_2{}^{2/3}$, or any other set of suitably defined functions and variables. Exercise 2 guides the reader through a choice that minimizes the algebraic manipulations. The resulting condition is

$$[v_2{}^{*2}(y_c)-v_1{}^{*2}(y_c{}^*)]\frac{dw^*}{dy_c{}^*}-g'w^*(y_c{}^*)F_2^2\frac{dh^*}{dy_c{}^*}=0\,, \qquad (5.2.17)$$

where y_c* denotes the position of the critical section. If *w** is constant, critical sections must occur at a point where $\partial h^*/\partial y^*=0$. In our previous, single-layer examples such points were generally restricted to sills. Later we will show that two-layer critical flow can also occur on a level part of the channel away from an obstacle. If *h** is constant but *w** varies, then critical flow can occur as before where $\partial w^*/\partial y^*=0$, as at a narrows, or where $v_1{}^{*2}=v_2{}^{*2}$. The latter possibility was first identified by Wood (1968) and the corresponding control section is called a *virtual control*. If the flow is unidirectional ($v_1^*v_2^*>0$) the shear velocity ($v_1{}^*-v_2{}^*$) is zero at such a control. A novel aspect of the virtual control is that it can occur where the channel width is changing, and we will later show that w^* must, in fact, be decreasing. The position y_c^* of the control depends on the flow itself and is not locked to a particular width.

An advantage of the Froude number plane representation is that critical flow lies along the diagonal line $F_1^2 + F_2^2 = 1$ (Figure 5.2.1). In the triangular region to the lower left of the diagonal the flow is subcritical. Above, the flow is supercritical. Some of the flow states in the supercritical range may be unstable with respect to long waves. The condition for stability (5.2.2) can be written in terms of the layer Froude numbers using (5.2.12) and the resulting threshold curve

$$[Q_r^{\ 1/3} F_1^{2/3} - F_2^{2/3}]^2 - Q_r^{\ 2/3} F_1^{-2/3} - F_2^{-2/3} = 0 \qquad (5.2.18)$$

is plotted in Figure 2a for Q_r=-1 (pure exchange flow). The threshold curve for Q_r=1 lies well above the critical diagonal and out of the range of the plot. An exchange flow state corresponding to any point lying above the (5.2.18) curve is formally unstable, though it remains to be seen whether such states are members of realizable solutions for reasonable upstream conditions.

The Froude number plane is not the only vehicle for representing solutions to two-layer flow. A reader seeking alternatives may wish to consult Dalziel (1991) or Baines (1995).

Exercises

1) Show that application of (5.2.15) to (5.2.13) and (5.2.14) leads to the critical condition G^2=1. (Hint: notice that F_1, F_2, and w^* only enter these relations in 2/3 power or 4/3 powers.)

2) Derive the regularity condition (5.2.17) as follows:

(a) Use the layer velocities v_1^* and v_2^* as independent variables and define functions $\mathcal{G}_1$ and $\mathcal{G}_2$ written solely in terms of these variables (and the geometric variables). This can be accomplished using equations (5.1.17), (5.2.6) and (5.2.11).

(b) Obtain (5.2.17) by evaluation of (1.5.11) and use of the functions defined in (a) and the two-layer critical condition.

3) Show that when critical flow occurs at a sill (where $dh^*/dy^*=0$, $d^2h^*/dy^{*2}<0$) that a supercritical to subcritical (or vice versa) transition must occur. That is, the flow cannot remain subcritical on either side of the sill.

Figure Captions

Figure 5.2.1 The critical diagonal and the long-wave stability threshold in the Froude number plane. (From Armi, 1986)

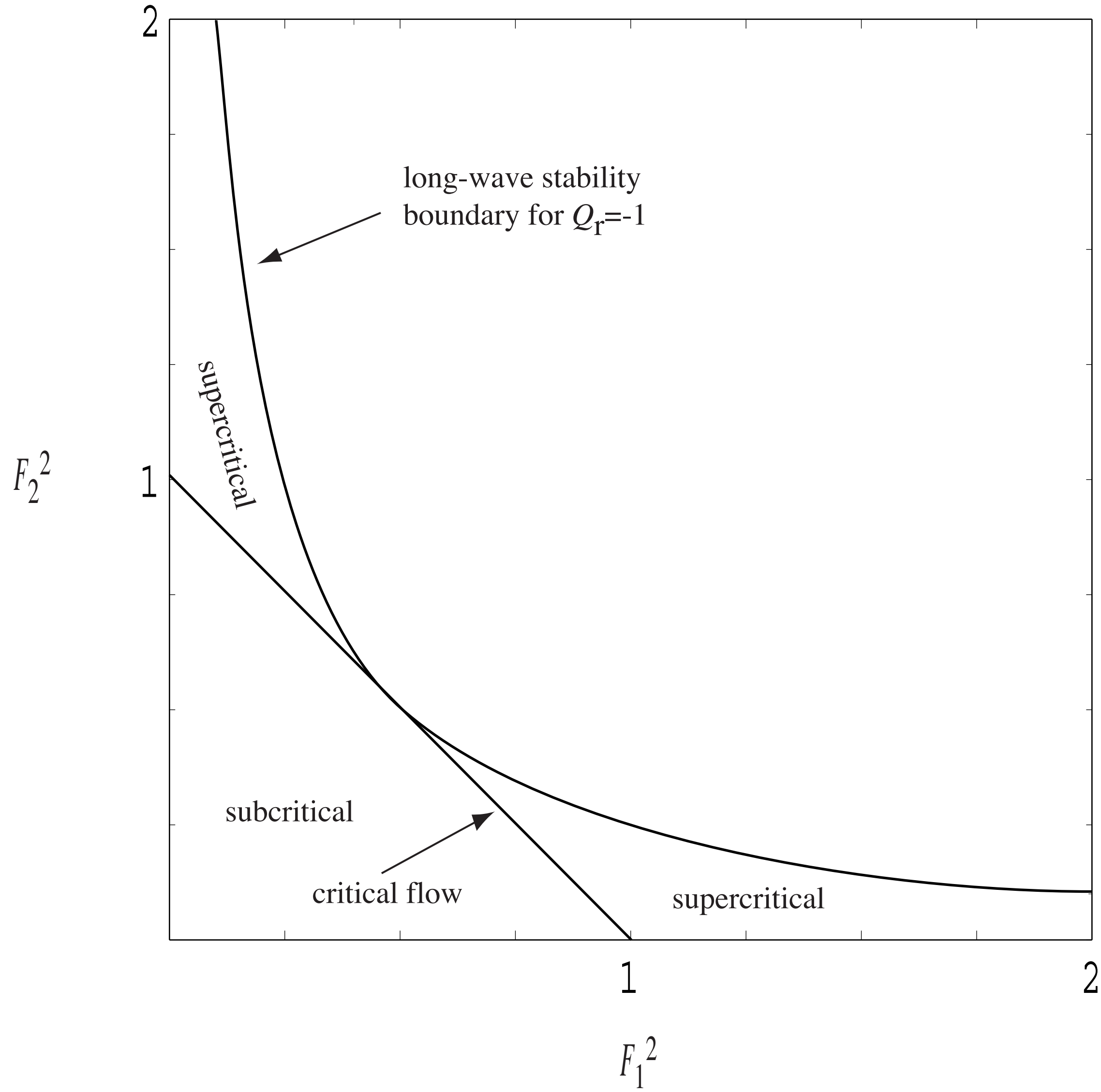
2
1
F_2^2
long-wave stability
boundary for Q_r=-1
supercritical
subcritical
critical flow
supercritical
1
2
F_1^2

Figure 5.2.1