Tellus (2006), 58A, 392–403
Printed in Singapore.

TELLUS

The abyssal stratification and circulation deduced from the principle of maximal entropy production

By MAHDI BEN JELLOUL* and RUI XIN HUANG, *Department of Physical Oceanography, Woods Hole Oceanographic Institution, Woods Hole, MA 02543, USA*

(Manuscript received 30 December 2004; in final form 21 November 2005)

ABSTRACT

A theory for the abyssal stratification and circulation is derived from the principle of maximal production of information entropy used in non-equilibrium statistical dynamics. For an abyssal ocean of simple geometry and simple, though plausible, parameterization of the vertical diffusivity, stratification profiles together with their associated circulation are obtained. These profiles are characterized by an abyssal thermocline thickness scaling as $\sqrt{\kappa_v H / w_{s0}}$, where w_{s0} is the upwelling velocity at the top of an abyssal ocean of depth H, and κ_v is its vertical diffusivity. Remarkably, the associated circulation is barotropic and coincides with the classical theory of the abyssal circulation by Stommel and Arons.

1. Introduction

Stommel and Arons (1960a, b) laid down the foundation for the classical theory of abyssal circulation; they used a homogeneous single-layer model to represent the abyssal ocean below the main thermocline. By assuming a uniform upwelling at the top of the abyssal ocean with a flat bottom, they deduced an abyssal circulation that includes a poleward interior flow and a strong western boundary current. Although the existence of deep western boundary currents has been confirmed by observations of oceans across the world, other aspects of the model assumptions and their dynamical consequences are questionable, and much effort has been devoted to enhance the theory of abyssal circulation. For example, in their classical model the abyssal circulation is treated as a single moving layer. The baroclinic structure of the abyssal flow was addressed successively by Pedlosky (1992) and by Edwards and Pedlosky (1995) who used a linear model to outline the role of the vertically distributed sources. Another idealization used in the model by Stommel and Arons (1960a, b) is that the upwelling velocity is horizontally uniform. Huang (1993) relaxed this assumption in a study of a nonlinear two-layer model where wind stress and buoyancy fluxes were both included. By coupling the wind-driven circulation with the thermally forced circulation, the interfacial upwelling rate is obtained as part of the solution.

*Corresponding author.
e-mail: benjello@univ-brest.fr
Current address: Mahdi Ben Jelloul, Laboratoire de Physique des Océans. UFR Sciences, 6 av. Le Gorgeu 29238 Brest Cedex 3, France.
DOI: 10.1111/j.1600-0870.2006.00177.x

One should note that, in these theories, the horizontally averaged stratification of the ocean interior is assumed a priori, and that the dynamic density is considered as perturbations to this basic stratification. Although these solutions provide interesting physical insight, an important question still remains: what physical processes set up the basic stratification?

By assuming a one-dimensional balance between upwelling and diapycnal mixing together with constant diapycnal diffusivity, Munk (1966) obtained an exponential density profile. This approach was extended to a one-and-a-half-dimensional model for the global ocean by Munk and Wunsch (1998): these authors calculated the global average stratification from the Levitus climatology and specified the globally integrated mass flux from other observations; they made estimates of the total mechanical energy required for sustaining the basic stratification and the renewal of deep water in the world oceans.

Understanding the maintenance of the stratification remains a great challenge to gain clear insight into the meridional circulation in the world oceans. In a pioneering paper, Sandström (1908) argued that the ocean is not a heat engine, and emphasized the need for an external source of mechanical energy to overcome the friction and dissipation associated with the circulation. He pointed out that in the case of a heating source placed at a pressure smaller than that of the cooling source, the system would be unable to find the mechanical energy required for sustaining the circulation, and thus unable to close the circulation by allowing an upwelling branch.

With a simple loop model, Huang (1999) illustrated the basic ideas associated with the Sandström theory. In addition, he proposed to employ sources of mechanical energy as an external parameter in investigations of the meridional overturning circulation. By using both scaling and a simple oceanic general circulation model, he showed that the circulation is determined by the distribution of a "mixing energy" associated with a set of climatic conditions and resulting from all mixing-producing processes. The state of the art in the mechanical energy balance for the world ocean circulation was recently discussed by Wunsch and Ferrari (2004) and Huang (2004).

In this study, we will use a variational principle to examine the connection between basic stratification and energy-supporting mixing. For a better understanding of the dynamical processes sustaining the general circulation we will split the ocean into two areas: the interior region and the convective one. The former is the place where the cold and dense deep water transported from the convective region is continuously pulled upward and converted into warm and light water in the upper ocean. This upwelling process is accomplished by diapycnal mixing through internal waves and turbulence; the external source of mechanical energy is simultaneously consumed and partly converted into the gravitational potential energy of the mean state.

On the other hand, the convective region is the place where deep water is formed; it thus initiates the renewal of water masses in the ocean. In this region, gravitational potential energy is lost to convective overturning. Although the mechanical energy of the ocean mean flow is also lost through friction (preliminary estimate for bottom drag yields 0.4 TW) and other dissipation mechanisms (1.1 TW are converted to eddy kinetic energy through baroclinic instability), convective adjustment due to cooling at high latitudes may be one of the most important sinks of mechanical energy in the global oceans: Huang and Wang (2003) give an estimate of 0.24 TW based on the monthly mean climatology but the value could be 10 times larger if the diurnal cycle is not filtered (all the cited estimates can be found in Huang, 2004).

One should note that the total area occupied by the convective region is much smaller than that of the interior region. Therefore, the actual flow of gravitational potential energy generated within the interior region and consumed in the convective one ensues from the specification of the source of mixing energy liable to generate gravitational potential energy in the ocean interior. The convective region is then somewhat slaved to the ocean interior since the characteristic adjustment timescale, defined as the ratio between total accumulated energy and energy fluxes, for the interior region is much larger than that of the convective region. Thus, in contrast to the classic school, which claims that the ocean is driven by push due to the formation of deep water, this study makes us suggest that the meridional overturning circulation is driven by pull created by the upwelling, and that the deep water renewal is induced by diapycnal mixing in the oceanic interior.

Our statement that the ocean is mainly pulled may sound controversial. However, we will propose a derivation where prescribing the abyssal gravitational potential energy together with some parameterization of mixing processes provides enough information to derive the abyssal circulation.

Our focus in this study is the relation between the stratification in the ocean interior and the external sources of mechanical energy. The convective region will be therefore treated as a component external to the interior region. Thus, the net source of deep water will be treated as external; we will show that it is linked to the prescribed global gravitational potential energy of the interior abyss, though its vertical distribution results from the interior advective-diffusive mass balance. In addition, the missing discussion of the energetics in a closed basin will make us omit the consumption of the gravitational potential energy through convection. Moreover, it is worth noting that our approach applies to circulation driven by diapycnal mixing (AABW and CDW circulations) where the isopycnals intersect the surface unlike in the case of the North Atlantic deep water which seem to be upwelled through Ekman suction in the Southern Ocean.

The specific approach used here is a variational method based on the principle of maximum entropy production. Even though this principle has been extensively discussed, the formulation by Jaynes (1979) has proven to be quite successful for applications associated with open non-equilibrium stationary states. Progresses along this line have been splendidly reviewed by Dewar (2003).

This paper is structured as follows: Section 2 is devoted to the model formulation and includes a brief discussion of the relevant balances that yield to the abyssal equations. The maximum production of entropy issued from the information theory is applied to a simple model of the abyssal ocean in Section 3. The application to the mean abyssal stratification is discussed in Section 4; in Section 4.1, a simple relation is assumed between the diapycnal mixing and stratification to derive the mean density profiles under the constraints of fixed global energy and mass. The discussion about the structure of the circulation associated with these solutions includes the horizontal circulation (Section 4.3) and its energy budget (Section 4.4). Section 5 emphasizes the connections with previous results, and discusses the natural extensions of this study.

2. Model formulations

2.1. Non-dimensional primitive equations

The model ocean is schematically depicted in Fig. 1. Our domain of interest is the abyssal ocean below the main thermocline (at $z^{\dim} = 0$) and above a flat bottom (at $z^{\dim} = -H$);

The horizontal extension we retain for our ocean is the classical apex geometry depicted in Fig. 2.

The constant horizontal area at depth $z^{\dim}$ is denoted by $A^{\dim}(z^{\dim}) = A^{\dim}$, and the total volume by $V^{\dim}$. The abyssal ocean receives water mass from the high latitudes depicted as the convective region on the right side in Fig. 1.

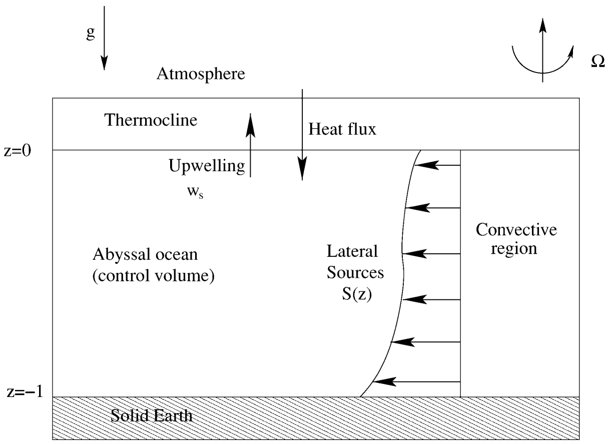

Fig. 1. Sketch of the abyssal circulation.

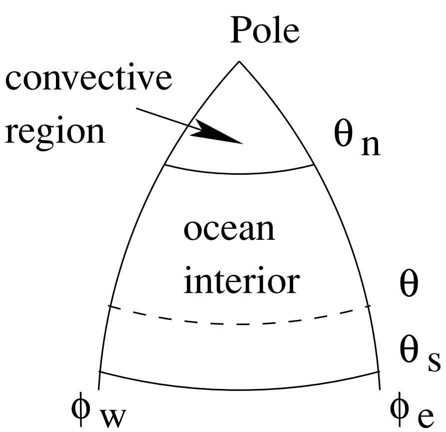

Fig. 2. Sketch of the horizontal extension of the model ocean. The control volume is delimited by the parallels θ_n and θ_s, and the meridians at longitudes ϕ_e and ϕ_w, respectively.

The density is separated into three components:

$$\rho^{\text{dim}} = \rho_0^{\text{dim}} + \rho_{\text{s}}^{\text{dim}}(z^{\text{dim}}) + \rho'^{\text{dim}}(x^{\text{dim}}, y^{\text{dim}}, z^{\text{dim}}), \tag{1}$$

where the first term on the right-hand side is the global mean density:

$$\rho_0^{\text{dim}} = \frac{\iiint_V \rho^{\text{dim}} \mathrm{d}V^{\text{dim}}}{V^{\text{dim}}}, \tag{2}$$

where the integral is carried over the total volume of the ocean; the second term on the right-hand side of (1) is the mean stratification defined as

$$\rho_{\text{s}}^{\text{dim}}(z) = \frac{\iint_A \rho^{\text{dim}} \mathrm{d}A^{\text{dim}}}{A^{\text{dim}}} - \rho_0^{\text{dim}}, \tag{3}$$

where the integral is carried over the horizontal area at the depth z^{dim}. The last term on the right-hand side of (1) is the density perturbation. By definition, the mean stratification density and the density perturbation both satisfy the following integral constraints:

$$\int_{-1}^{0} \rho_{\text{s}}^{\text{dim}} \mathrm{d}z^{\text{dim}} = 0, \qquad \iint_{A^{\text{dim}}} \rho'^{\text{dim}} \mathrm{d}A^{\text{dim}} = 0. \tag{4}$$

Introducing two density scales:

$$\Delta\rho_{\text{s}} = \rho_{\text{s}}^{\text{dim}}\big|_{z^{\text{dim}}=-1} - \rho_{\text{s}}^{\text{dim}}\big|_{z^{\text{dim}}=0}, \tag{5}$$

$$\Delta\rho' = \max|\rho'^{\text{dim}}|, \tag{6}$$

allows us to write the density field in terms of the non-dimensional stratification and dynamical densities:

$$\rho^{\text{dim}} = \rho_0^{\text{dim}} + \Delta\rho_{\text{s}}[\rho_{\text{s}}(z) + \lambda\rho'(x, y, z, t)] \tag{7}$$

where the non-dimensional parameter $\lambda = \Delta\rho'/\Delta\rho_{\text{s}}$ represents the relative contribution due to the dynamical density field ρ' compared to the mean stratification density ρ_{s}. Let us also introduce the non-dimensional coordinates $(x, y, z) = (x^{\text{dim}}/R, y^{\text{dim}}/R, z^{\text{dim}}/H)$ where R is the earth's radius used as a characteristic horizontal scale. The hydrostatic relation is split into two components as

$$\frac{\partial p_{\text{s}}}{\partial z} + \rho_{\text{s}} = 0, \quad \frac{\partial p'_{\text{bc}}}{\partial z} + \rho' = 0, \tag{8}$$

where the non-dimensional dynamical pressure perturbation p' is defined as

$$p' = p'_{\text{bt}} + p'_{\text{bc}} = \frac{p^{\text{dim}} - \int_{-H}^{z^{\text{dim}}} \left(\rho_{\text{s}}^{\text{dim}} g z^{\text{dim}} - \rho_0^{\text{dim}} g z^{\text{dim}}\right) \mathrm{d}z^{\text{dim}}}{g\Delta\rho' H}, \tag{9}$$

where the subscripts bt and bc denote the barotropic and baroclinic components, respectively, and $g$is the acceleration of gravity.

Let us introduce the Burger number:

$$\text{Bu} = \frac{R_{\text{d}}^2}{L^2}, \quad R_{\text{d}} = \frac{\sqrt{g'H}}{2\Omega}, \quad g' = \frac{g\Delta\rho_{\text{s}}^{\text{dim}}}{\rho_0^{\text{dim}}} = (N^{\text{dim}})^2 H, \tag{10}$$

where R_{d} is the Rossby radius of deformation, g' is the reduced gravity, and N^{dim} is the buoyancy frequency. The Rossby number is defined as

$$\text{Ro} = \frac{U}{2\Omega R} = \frac{V}{2\Omega R} = \frac{W}{2\Omega H}, \tag{11}$$

where U, V and W are the scales of the horizontal and vertical velocities, respectively, and Ω denotes the earth's rotation rate. The scale of vertical and horizontal velocities is related through the continuity equation; the time scale is $1/2\Omega$. Finally, the non-dimensional diffusivities resulting from the density mixing processes are defined as

$$\kappa_{\text{h}} = \frac{\kappa_{\text{h}}^{\text{dim}}}{2\Omega R^2}, \quad \kappa_v = \frac{\kappa_v^{\text{dim}}}{2\Omega H^2}, \tag{12}$$

and the non-dimensional eddy viscosity is:

$$\upsilon = \frac{\upsilon^{\mathrm{dim}}}{2\Omega R^2}. \tag{13}$$

The corresponding non-dimensional primitive equations are then:

$$\frac{\partial \mathbf{u}_\mathrm{h}}{\partial t} + \mathrm{Ro}\mathbf{u}\cdot\nabla\mathbf{u}_\mathrm{h} + \sin\theta\mathbf{k}\times\mathbf{u}_\mathrm{h} + \frac{\lambda\mathrm{Bu}}{\mathrm{Ro}}\nabla_\mathrm{h}(p'_\mathrm{bc}+p'_\mathrm{bt}) = \nabla_\mathrm{h}(\upsilon\nabla_\mathrm{h}\mathbf{u}_\mathrm{h}) + \left(\frac{R}{H}\right)^2\frac{\partial}{\partial z}\left(\upsilon\frac{\partial \mathbf{u}_\mathrm{h}}{\partial z}\right) \tag{14a}$$

$$\frac{\partial p_s}{\partial z} + \lambda\frac{\partial p'_{bc}}{\partial z} + \rho_\mathrm{s} + \lambda\rho' = 0 \tag{14b}$$

$$\lambda^{-1}\frac{\partial\rho_\mathrm{s}}{\partial t} + \frac{\partial\rho'}{\partial t} + \mathrm{Ro}\mathbf{u}\cdot\nabla\rho' + \mathrm{Ro}\lambda^{-1}w\frac{\partial\rho_\mathrm{s}}{\partial z} + \mathrm{Ro}w\frac{\partial\rho'}{\partial z} = \lambda^{-1}\frac{\partial}{\partial z}\left(\kappa_\mathrm{v}\frac{\partial\rho_\mathrm{s}}{\partial z}\right) + \nabla_\mathrm{h}(\kappa_\mathrm{h}\nabla_\mathrm{h}\rho') + \frac{\partial}{\partial z}\left(\kappa_\mathrm{v}\frac{\partial\rho'}{\partial z}\right) \tag{14c}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{14d}$$

where ∇ and ∇_h indicate the three-dimensional and horizontal gradient operators, $\mathbf{u} = (u, v, w)$ and $\mathbf{u}_\mathrm{h} = (u, v)$ are the non-dimensionnal three-dimensional and horizontal velocity vectors, respectively, and θ is the geographical latitude. One should note that these equations need no convective adjustment parameterization because the convective region is not contained in the control volume. In the following discussion, viscous processes in the momentum equations will be omitted.

2.2. *Energy budget*

Since our model is based on the Boussinesq approximation, the continuity equation is reduced to volume conservation, and thus the internal and mechanical energies are decoupled. The mechanical energy regulates the motion. Furthermore, the kinetic energy being much smaller than the gravitational potential energy, the constraint on gravitational potential energy is the most relevant quantity for our study. The non-dimensional total gravitational potential energy expressed by

$$\chi = \iiint_V \rho z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{15}$$

can be separated into two components:

$$\chi_0 = \iiint_V \rho_0\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{16}$$

$$\chi_s = \iiint_V \rho_s\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = A\int_{-1}^{0}\rho_s z\,\mathrm{d}z, \tag{17}$$

where χ_0 represents the inert part of the gravitational potential energy and χ_s is called the stratified gravitational potential energy, which is a convenient measure of the energetic impact of stratification of the abyssal ocean under the constraint of a constant mass: any shift in the value of ρ_0 would change the global potential energy, but without impacting the stratification potential energy. Moreover, one should note that the integrated gravitational potential energy related to the dynamical density ρ' is zero by definition.

The equation governing the gravitational potential energy evolution reads

$$\frac{\partial\chi}{\partial t} = \frac{\partial\chi_\mathrm{s}}{\partial t} = -\mathrm{Ro}\int_{-1}^{0}\oint_{\partial A(z)}\rho z\mathbf{u}_\mathrm{h}\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}z + \iiint_V z\left[\frac{\partial}{\partial z}\left(\kappa_v\frac{\partial\rho}{\partial z}\right) + \nabla_\mathrm{h}(\kappa_\mathrm{h}\nabla_\mathrm{h}\rho)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \mathrm{Ro}\iiint_V p\frac{\partial w}{\partial z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \mathrm{Ro}\left[\iint_{A(z)} w(\rho+\rho z)\,\mathrm{d}x\,\mathrm{d}y\right]_{z=-1}^{z=0} \tag{18}$$

where $\partial A(z)$ denotes the horizontal boundary of the model ocean at the depth z, ds is the curvilinear abscissa along this boundary, and $\mathbf{n}$ is the unit vector normal to the surface boundary; the inner integral in the first term in the right-hand side of (18) has to be taken along the horizontal periphery of the model at the depth z. The first term on the right-hand side is a sink of gravitational energy when deep water is formed in the convective region. The second term on the right-hand side of (18) is the source due to mixing, and in the case of a horizontally homogeneous vertical diffusivity it is reduced to

$$\iiint z\frac{\partial}{\partial z}\left(\kappa_\mathrm{v}\frac{\partial}{\partial z}(\rho_\mathrm{s}+\lambda\rho')\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{-1}^{0} Az\frac{\partial}{\partial z}\left(\kappa_\mathrm{v}\frac{\partial\rho_\mathrm{s}}{\partial z}\right)\,\mathrm{d}z = \int_{-1}^{0} A\left(\kappa_\mathrm{v}N^2 - \kappa_\mathrm{v}N^2|_{z=-1}\right)\,\mathrm{d}z, \tag{19}$$

where N^2 is the non-dimensional buoyancy frequency defined as:

$$N^2(z) = -\frac{\partial\rho_\mathrm{s}}{\partial z}. \tag{20}$$

The term for the conversion of kinetic to potential energy is represented by both the third term on the right-hand side of (18) and the first term in the last surface integral; the latter describes the pressure work at the horizontal surface. The last surface term is the gravitational potential energy flux at the horizontal interface.

The total kinetic energy:

$$E_\mathrm{k} = \frac{\mathrm{Ro}^2}{\mathrm{Bu}}\iiint_V \frac{\mathbf{u}_\mathrm{h}^2}{2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{21}$$

satisfies the following evolution equation:

$$\frac{\partial E_\mathrm{k}}{\partial t} = -\iint_{\partial V}\left[\left(\frac{\mathrm{Ro}^2}{\mathrm{Bu}}\frac{\mathbf{u}_\mathrm{h}^2}{2} + \mathrm{Ro}\cdot p\right)\mathbf{u} - \upsilon\frac{\mathrm{Ro}^2}{\mathrm{Bu}}\nabla\frac{\mathbf{u}_\mathrm{h}^2}{2}\right]\cdot\mathbf{n}\,\mathrm{d}S + \iiint_V\left(-\mathrm{Ro}\cdot w\rho - \upsilon\frac{\mathrm{Ro}^2}{\mathrm{Bu}}|\nabla\mathbf{u}_\mathrm{h}|^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{22}$$

where ∂V and dS denote the surface delimiting the control volume and the surface element with unit vector **n,** respectively.

2.3. Regime parameters

For basin scale flows, we have the following typical values:

$$a = 6.4 \times 10^6\,\mathrm{m}, \quad 2\Omega = 1.4 \times 10^{-4}\mathrm{s}^{-1},$$
$$H = 4 \times 10^3\,\mathrm{m}, \quad \rho_0 \simeq 10^3\,\mathrm{kg\,m}^{-3}, \quad g = 9.81\,\mathrm{m\,s}^{-1}$$
$$\kappa_{\mathrm{v}}^{\mathrm{dim}} \simeq 10^{-4}\,\mathrm{m}^2\mathrm{s}^{-1}, \quad N^{\mathrm{dim}} \simeq 10^{-3}\mathrm{s}^{-1}, \quad w^{\mathrm{dim}} \simeq 10^{-7}\,\mathrm{m\,s}^{-1}. \tag{23}$$

No value for the eddy viscosity is given since viscous dissipation is unimportant for this study and will always be neglected. Observations (Ledwell et al., 1993) have indicated that, away from rough topography, the diffusivity κ_v^{dim} is on the order of $10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$; however, we keep the relatively high value of basin-mean κ_v^{dim} compatible with mean stratification as computed by Munk and Wunsch (1998). The non-dimensional numbers are thus

$$\mathrm{Bu} \simeq 10^{-5}, \quad \mathrm{Ro} \simeq 10^{-7}, \quad \kappa_{\mathrm{v}} \simeq 10^{-7} \tag{24}$$

Geostrophy requires:

$$\frac{\lambda \mathrm{Bu}}{\mathrm{Ro}} = O\,(1)\,, \quad \mathrm{Ro} \ll 1, \tag{25}$$

where we assume that the Rossby number is a small parameter; it, thus, allows us to perform a weakly nonlinear expansion of the system. To separate the mean density profile from the dynamical density, the relative isopycnal displacement is further assumed to be $\lambda \ll 1$. This assumption implies that the horizontal fluxes of buoyancy are dominated by the horizontal advection of the mean buoyancy.

The desire to solve the equation order by order and to simplify the algebra makes us choose

$$\mathrm{Bu} = O(\varepsilon)\,, \quad \lambda = O(\varepsilon)\,, \quad \mathrm{Ro} = O(\varepsilon^2), \tag{26}$$

where ε is a small parameter.

This scaling is reminiscent of large-scale quasi-geostrophic theories reviewed in Pedlosky (1996) since it allows the geostrophic balance at first order.

Now, let us proceed to the scaling of diffusivity. By assuming that the primary balance in the buoyancy equation lies between vertical advection and vertical mixing, the buoyancy eq. (14c) yields

$$\mathrm{Ro}\lambda^{-1} w \frac{\partial \rho_{\mathrm{s}}}{\partial z} \simeq \lambda^{-1} \frac{\partial}{\partial z}\left(\kappa_{\mathrm{v}} \frac{\partial \rho_{\mathrm{s}}}{\partial z}\right). \tag{27}$$

Since N^2 is of order one, horizontal averaging (27) leads to

$$\mathrm{Ro}\frac{\iint_A w\,\mathrm{d}x\mathrm{d}y}{A} \simeq \frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa_{\mathrm{v}} N^2\right), \tag{28}$$

and suggests $Ro = O(\kappa_v)$. However, a careful examination of (28) reveals that the vertical scale of the mean stratification N^2 controls the intensity of the horizontally averaged upwelling for a given vertical diffusivity. For example, if N^2 and κ_v are both depth independent, this upwelling vanishes at first order. The latter situation was explored by Pedlosky (1996, p. 437) who used the relation

$$\mathrm{Ro}\lambda^{-1} w \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z} \simeq \frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa_{\mathrm{v}} \frac{d\rho'}{dz}\right), \tag{29}$$

which can be seen (considered) as the departure from the horizontal average (28). The associated baroclinic flow can be calculated by using geostrophy. In addition, if we require that this baroclinic flow is comparable to a barotropic component *a la* Stommel and Arons (1960a, b), we obtain the following non-dimensional balance

$$\frac{\mathrm{Bu}}{\kappa_{\mathrm{v}}} \simeq 70 \simeq O\,(1)\,. \tag{30}$$

Thus, it is then natural to assume:

$$\kappa_{\mathrm{v}} = \varepsilon \kappa(z), \tag{31}$$

where $\kappa(z)$ is a non-dimensional vertical diffusivity profile of order one.

As discussed below, the main vertical stratification is controlled by the fluxes at the base of the thermocline and lateral sources. It is worth recalling that the diffusive density fluxes associated with the mean stratification are not a priori larger than the one associated with the perturbation. For example, let us consider a stable linear density profile with constant buoyancy frequency; for a constant vertical diffusivity, the associated buoyancy flux is constant, so the diapycnal velocity is zero. This, however, does not mean that the dynamical (horizontally varying) isopycnal displacements are large. We can thus assume that the two components of the density field induce diapycnal velocity of presumably comparable magnitude and retain the scaling (31). One should keep in mind that, here, the Rossby number Ro is not a direct measure of the intensity of the horizontally averaged upwelling because it can be reduced to zero according to (28); but it is a scaling of its departure from horizontal homogeneity.

Scaling (26, 30, 31) does not exactly match the values (24), but allows us to recover the simplicity of the derivation of Edwards and Pedlosky (1995) though the horizontal advection of buoyancy is neglected in their study. To sum up, our scaling represents a compromise between the actual values of non-dimensional parameters and simple balances; moreover, though it neglects isopycnal eddy motions and focuses on vertical diapycnal mixing, it is simple enough for a tractable asymptotic expansion.

2.4. Abyssal equations

At the first order in ε, the steady state of the system (14) verifies

$$-\frac{\sin 2\theta}{2} v + \frac{\lambda \mathrm{Bu}}{\mathrm{Ro}} \frac{\partial p}{\partial \phi} = 0, \tag{32a}$$

$$\sin\theta u + \frac{\lambda \mathrm{Bu}}{\mathrm{Ro}}\frac{\partial p}{\partial \theta} = 0, \tag{32b}$$

$$\frac{\partial p}{\partial z} + \rho = 0, \tag{32c}$$

$$-(\tan\theta)^{-1} v + \frac{\partial w}{\partial z} = 0, \tag{32d}$$

where ϕ is the longitude.

The barotropic component is obtained by integrating (32d):

$$v_{\mathrm{bt}} = \int_{-1}^{0} v\,\mathrm{d}z = \tan\theta\,[w_{s0} + w_e(\phi,\theta)], \tag{33}$$

where the vertical velocity at the top of the abyss is separated into two components: a horizontal mean $w_{s0} = w_s|_{z=0}$ and an "Ekman pumping" w_e, which has a zero horizontal mean by definition. In addition, to derive (34) we assume a vanishing of vertical velocity at the bottom. The "Ekman pumping" can be replaced by the vertical velocity at the base of the main thermocline, if the ocean control volume is defined as being below the main thermocline.

The mean upwelling velocity is defined as the basin-horizontal average:

$$w_s(z) = \frac{H}{\kappa_v^{\mathrm{dim}} A^{\mathrm{dim}}} \iint_A w^{\mathrm{dim}}\,\mathrm{d}x\,\mathrm{d}y = \frac{\mathrm{Ro}}{\varepsilon A}\iint_A w\mathrm{d}x\mathrm{d}y. \tag{34}$$

Horizontally averaging (27) yields to

$$w_s = \frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right)\left(\frac{\mathrm{d}\rho_s}{dz}\right)^{-1} = \kappa\frac{\mathrm{d}}{\mathrm{d}z}\left[\ln\left|\kappa\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right|\right]. \tag{35}$$

The volume conservation (incompressibility) leads to an integral constraint between the lateral sources (Fig. 1):

$$S(z) = \oint_{\partial A(z)} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s, \tag{36}$$

and to the horizontally averaged upwelling:

$$\frac{\mathrm{d}w_s}{\mathrm{d}z} = \frac{S(z)}{A}. \tag{37}$$

3. Principle of maximum entropy production

Determining the mean stratification involves finding not only $\rho_s(z)$, but also $w_s(z)$, $\kappa(z)$ and the distribution of lateral sources $S(z)$ because these variables are all dynamically linked. Therefore, in addition to eqs. (35) and (37), we need two more equations. A parameterization of diffusivity as a function of the flow properties (Section 4.1), essentially stratification, provides one equation; the one still missing can be obtained from a variational principle on condition the mean stratification maximizes the entropy production rate.

3.1. Conservation laws

In a recent study, Dewar (2003) applied the Jaynes' (1979) information theory formalism of statistical mechanics to the stationary states of open, non-equilibrium systems. The analysis developed hereafter follows, step by step, his approach. For the case discussed here, the primitive eq. (14) are expressed as the following conservation laws:

a) The energy conservation:

$$\frac{\partial e}{\partial t} = -\nabla\cdot\mathbf{f}_e + q_e, \tag{38a}$$

where:

$$e = \frac{\mathrm{Ro}^2}{\mathrm{Bu}}\frac{\mathbf{u}_h^2}{2} + \rho z, \tag{38b}$$

$$\mathbf{f}_e = \left(\mathrm{Ro}p + \frac{\mathrm{Ro}^3}{\mathrm{Bu}}\frac{\mathbf{u}_h^2}{2} + \mathrm{Ro}\,\rho z\right)\mathbf{u} - \nu\frac{\mathrm{Ro}^2}{\mathrm{Bu}}\nabla\frac{\mathbf{u}_h^2}{2} - \left(\kappa_v\frac{\partial\rho}{\partial z}z\right)\mathbf{k}, \tag{38c}$$

$$q_e = -\nu\frac{\mathrm{Ro}^2}{\mathrm{Bu}}|\nabla\mathbf{u}_h|^2 - \kappa_v\frac{\partial\rho}{\partial z}, \tag{38d}$$

b) The buoyancy conservation:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{f}_\rho, \tag{39a}$$

where:

$$\mathbf{f}_\rho = \rho\mathbf{u} - \kappa_v\frac{\partial\rho}{\partial z}\mathbf{k}, \tag{39b}$$

where $\mathbf{k}$ is the vertical unit vector.

A major difference with Dewar (2003) is that we consider a globally constrained, stationary system instead of a stationary system with a locally prescribed initial state.

3.2. Notations and goal

Following Dewar (2003) requires the introduction of some compact notations: Γ denotes the "microscopic" phase space trajectories during a time interval τ, and p_Γ is their probability distribution. In addition, $\mathbf{d} = (e,\rho)$ is the macroscopic state vector, the fluxes on the boundaries are $\mathbf{F} = (\mathbf{f}_e, \mathbf{f}_\rho)$, and the sources are $\mathbf{Q} = (q_e, q_\rho) = (q_e, 0)$.

The primitive eqs. (38) and (39) can then be rewritten in the following compact form:

$$\frac{\partial \mathrm{d}}{\partial t} = -\nabla\cdot\mathbf{F} + \mathbf{Q}. \tag{40}$$

For a quantity X, $\overline{X} = \frac{1}{\tau}\int_0^\tau X(t)\,\mathrm{d}t$ denotes its time average over the time interval τ, and $\langle X\rangle = \sum_\Gamma p_\Gamma X_\Gamma$ denotes its expectation value over the probability of the underlying microscopic paths, where X_Γ is the value of X for path Γ.

The main goal of our analysis is to maximize the path information entropy:

$$S_I = -\sum_\Gamma p_\Gamma \ln p_\Gamma, \tag{41}$$

with respect to Γ, subject to external constraints (boundary conditions), stationarity constraints (steady state, global energy, and mass balance) and other a priori information describing the system (incompressibility), i.e. the conservation laws discussed in Section 3.1.

3.3. Path distribution

Let us, first, apply the following constraints to p_Γ:

$$\sum_\Gamma p_\Gamma = 1, \qquad \text{normalization}, \tag{42a}$$

$$\sum_\Gamma p_\Gamma \int_V \mathrm{d}(\mathbf{r};0) = \left\langle \int_V \mathrm{d}(\mathbf{r};0) \right\rangle, \quad \text{initial global energy and mass}, \tag{42b}$$

where the integral is carried over the control volume; moreover, to lighten the notation, multiple integral symbol and integrand are omitted.

Maximizing the information entropy with respect to p_Γ, subject to the constraints (42), yields the distribution as:

$$p_\Gamma = \frac{e^{A_\Gamma}}{Z}, \tag{43}$$

where $Z = \sum_\Gamma e^{A_\Gamma}$ is the partition function, and A_Γ is the path action given by:

$$A_\Gamma = \lambda \cdot \int_V \mathrm{d}(\mathbf{r};0)_\Gamma = \lambda_e \int_V e(\mathbf{r};0)_\Gamma + \lambda_\rho \int_V \rho(\mathbf{r};0)_\Gamma, \tag{44}$$

where $\lambda = (\lambda_e, \lambda_\rho)$ are the Lagrange multipliers associated with the conservation of the global mass, global energy, and the fixed fluxes of these quantities at the boundaries of the model ocean. In order to introduce the constraints imposed on the flow by the conservation (38,39), we use the definition of the overbar as a time average to write:

$$\mathrm{d}(\mathbf{r};0)_\Gamma = \frac{\mathrm{d}(\mathbf{r};0) + \mathrm{d}(\mathbf{r};\tau)_\Gamma}{2} - \frac{\tau \overline{\frac{\partial \mathrm{d}(\mathbf{r};t)_\Gamma}{\partial t}}}{2}. \tag{45}$$

After integration by parts, the path action (43) becomes:

$$A_\Gamma = \frac{\lambda}{2} \cdot \int_V [\mathrm{d}(\mathbf{r};0)_\Gamma + \mathrm{d}(\mathbf{r};\tau)_\Gamma] - \frac{\tau\lambda}{2} \cdot \int_V \overline{\mathbf{Q}}(\mathbf{r})_\Gamma + \frac{\tau\lambda}{2} \cdot \int_{\partial V} \overline{\mathbf{F}^{\mathrm{n}}}(\mathbf{r})_\Gamma \cdot \mathbf{n}, \tag{46}$$

where $\mathbf{n}$ indicates the normal direction of the control volume surface ∂V over which the surface integral is carried, and the superscript n denotes the normal component of the flux.

The action can be rewritten as:

$$A_\Gamma = \frac{\lambda_e}{2} \int_V [e(\mathbf{r};0)_\Gamma + e(\mathbf{r};\tau)_\Gamma] + \frac{\lambda_\rho}{2} \int_V [\rho(\mathbf{r};0)_\Gamma + \rho(\mathbf{r};\tau)_\Gamma] + \frac{\sum_\Gamma}{2}, \tag{47}$$

where the dimensionless entropy production is now expressed as:

$$\Sigma_\Gamma = -\tau\lambda_e \cdot \int_V \overline{q_e}(\mathbf{r})_\Gamma + \tau\lambda_e \int_{\partial V} \overline{\mathbf{f}^{\mathrm{n}}_e}(\mathbf{r})_\Gamma \cdot \mathbf{n} + \tau\lambda_\rho \int_{\partial V} \overline{\mathbf{f}^{\mathrm{n}}_\rho}(\mathbf{r})_\Gamma \cdot \mathbf{n}, \tag{48}$$

When assuming a steady state, the first two integrals in (47) are symmetric with regard to the time reversal $\tau \to -\tau$. The entropy production (48) reflects the irreversibility processes since it is not symmetric with regard to the time reversal. If we now additionally assume fixed global mass and energy, we can anticipate that the probability of the steady states will depend only on their entropy production.

3.4. Mean field approach

Let us, now, adopt the classical mean field approach (the thermodynamic limit) and neglect the fluctuations about the mean behavior. The statistical mean is, then, the most probable (steady) state. Dewar (2003) showed that maximizing the information entropy (41) is then equivalent to maximizing the mean (over the different paths, Γ) entropy production rate with respect to the Lagrange multipliers λ, with the assumptions of stationarity, conservation of global mass and global energy and fixed fluxes at the boundaries. After ensemble-averaging over the set of paths, the entropy production variation is:

$$\begin{aligned}
\delta \sum &= -\tau\delta\lambda \cdot \int_V \langle\overline{Q}\rangle - \tau\lambda \cdot \int_V \delta\langle\overline{Q}\rangle + \tau\delta\lambda \cdot \int_{\partial V} \langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle \cdot \mathbf{n} + \tau\lambda \cdot \int_{\partial V} \delta\langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle \cdot \mathbf{n} \\
&= -\tau\delta\lambda \cdot \left[\int_V \langle\overline{Q}\rangle - \int_{\partial V} \langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle\right] - \tau\lambda \cdot \int_V \delta\langle\overline{Q}\rangle + \tau\lambda \cdot \delta \int_{\partial V} \langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle \\
&= -\tau\lambda \cdot \int_{\partial V} \delta\langle\overline{Q}\rangle + \tau\lambda \cdot \int_{\partial V} \delta\langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle,
\end{aligned} \tag{49}$$

In deriving (49) we use the global equilibrium relation:

$$\int_V \langle\overline{Q}\rangle - \int_{\partial V} \langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle = 0. \tag{50}$$

Finding the extrema of entropy production is thus equivalent to solving the following variational equation:

$$\delta \sum_\Gamma -\alpha\delta E - \beta\delta M - \gamma \int_{\partial V} \delta\langle\overline{\mathbf{F}^{\mathrm{n}}}\rangle = 0, \tag{51}$$

where the global mean energy and mass are defined as:

$$E = \int_V \langle\overline{e}\rangle, \quad M = \int_V \langle\overline{\rho}\rangle \tag{52}$$

This variational eq. (51) is subject to constraints of constant net fluxes of mass (zero net flux) and energy (to compensate the production of energy by diffusion). Let us use the following values for the Lagrange multipliers:

$$\alpha = -\tau\lambda_e\eta, \quad \beta = \tau\lambda_\rho\mu, \tag{53}$$

Relation (51) can then be rewritten as:

$$\delta \int_V \left\langle \overline{\left(\kappa_v N^2 - \nu \frac{Ro^2}{Bu}|\nabla \mathbf{u}|^2\right)} \right\rangle - \eta\delta E + \mu\delta M = 0, \quad \gamma = \tau\lambda, \tag{54}$$

where the first term appears as a classical entropy production term that should vanish in the event of missing irreversible processes such as diffusion and dissipation. With neither diffusion, relation (54) would be a minimum dissipation theorem. Because of the existence of diffusive processes in the ocean and according to relation (54), the abyssal ocean would organize its stratification and associated circulation so as to maximize its production of entropy for a given mass and energy. One should note that, here, the production of entropy coincides with that of energy.

In the following analysis the time mean solution is assumed to coincide with the steady one. Thus, for simplification purpose, the time average bar and ensemble-average bracket are both dropped. Moreover, the effect of dissipation is neglected, and focus is on the vertical diffusion.

4. Abyssal stratification and circulation subject to maximizing entropy production

4.1. Parameterization of diapycnal diffusivity

Observations and bulk estimates (Polzin et al., 1997; Munk and Wunsch, 1998) have indicated that turbulent mixing in the oceans is far from being homogeneous. Thus, in departure from a constant diffusivity, diapycnal mixing can be parameterized in terms of a power law of the stratification:

$$\kappa_v = aN^m, \tag{55}$$

where a is assumed constant over the whole water column although this might be a strong assumption, and $m \leq 0$; mixing is thus a decreasing function of stratification. Such a power law dependence has been supported by local observations, especially in fjords (Stigebrandt, 1979). Gargett (1984) suggested a value of $m \simeq -1.2$ in the open ocean whereas Stigebrandt and Aure (1989) reported a value of $m \simeq -1.5$ from measurements in fjords.

4.2. The maximal entropy solution

This section will focus on the search for a solution satisfying (54) while maximizing the production of entropy under fixed global energy and global mass constraints:

$$\chi_s = \iiint_V \rho_s z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = A\int_{-1}^{0} \rho_s z\,\mathrm{d}z = E_s, \tag{56}$$

$$M = \iiint_V \rho_s \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = A\int_{-1}^{0} \rho_s\,\mathrm{d}z = 0. \tag{57}$$

One should note the use of the gravitational potential energy as a constraint instead of the total energy; this choice relies on the fact that the kinetic energy is much smaller than the potential one; so, at first order, the energy consists in the gravitational potential energy. Furthermore, the stratified gravitational potential energy is considered only because, χ_0, the inert component of gravitational potential energy, is invariant when the total mass remains unchanged. Thus, the variational eq. (54) is reduced to

$$\delta\sigma - \eta\delta\chi + \mu\delta M = 0, \tag{58}$$

where the entropy production (we neglect viscosity) is

$$\sigma = A\int_{-1}^{0} \kappa N^2\,\mathrm{d}z. \tag{59}$$

Under assumption (55), the entropy variation is

$$\delta\sigma = A\int_{-1}^{0} \frac{a\,(m+2)}{2}\frac{dN^m}{\mathrm{d}z}\delta\rho\,\mathrm{d}z - A[\kappa\delta\rho]_{z=-1}^{z=0}, \tag{60}$$

where the boundary terms can be eliminated when a fixed density is assumed at the top/bottom of the abyssal ocean. This corresponds to a rearrangement of the fluid layers thickness between two fixed isopycnals to maximize the entropy production under constraints of fixed mass and energy.

For diapycnal diffusivities in forms of (55), equation (58) is reduced to

$$\frac{a\,(m+2)}{2}\frac{\mathrm{d}N^m}{\mathrm{d}z} - \eta z + \mu = 0. \tag{61}$$

By using (55) and (36), the vertical velocity can be rewritten as

$$w_s = \frac{a\,(m+2)}{m}\frac{\mathrm{d}N^m}{\mathrm{d}z}. \tag{62}$$

and using (61) gives

$$w_s = \frac{2\,(\eta z - \mu)}{m} = w_{s0}\,(z+1)\,, \tag{63}$$

where the boundary condition used is $w_s = 0$ at $z = -1$. In addition, by setting

$$\eta = \frac{m w_{s0}}{2}, \tag{64}$$

the vertical velocity at the upper surface is equal to w_{s0}. The profile of the buoyancy frequency can then be written as

$$N = \left\{(N|_{z=-1})^m + \left[[(N|_{z=0})^m - (N|_{z=-1})^m\right](1+z)^2\right\}^{1/m}, \tag{65a}$$

$$w_s = \frac{2a_0\,(2+m)\left[(N|_{z=0})^m - (N|_{z=-1})^m\right](1+z)}{m}. \tag{65b}$$

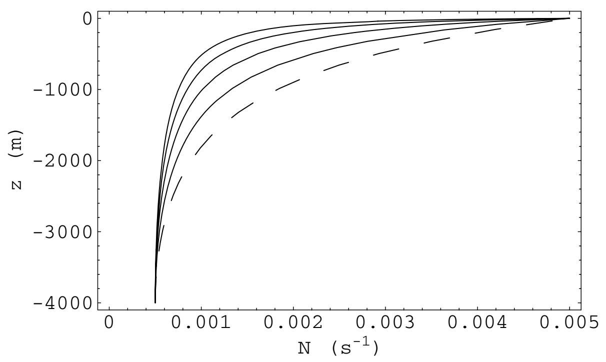

Fig. 3. Various profiles of the buoyancy frequency N^{dim} as a function of z^{dim}, assuming $N^{\mathrm{dim}} = 5 \times 10^{-4}\ \mathrm{s}^{-1}$ at the bottom and $N^{\mathrm{dim}} = 5 \times 10^{-3}\ \mathrm{s}^{-1}$ at the top of a 4000 m abyssal ocean. The solid curves correspond to $m = -1.9, -1.5, -1, -0.5$, with $m = -1.9$ at the top; the dashed curve corresponds to the constant diffusivity case, $m = 0$.

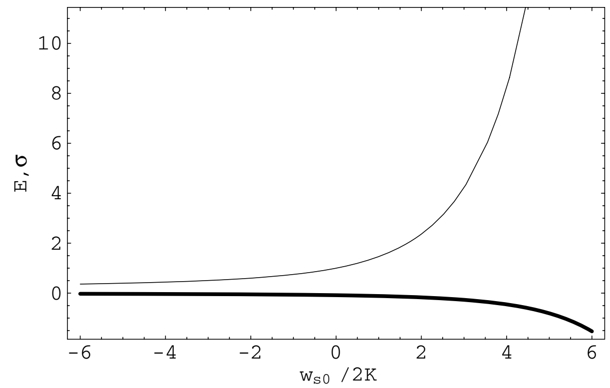

Fig. 4. Energy and entropy for different values of the upwelling velocity with a fixed bottom stratification (constant diffusivity case).

When assuming $N|_{z=0} > N|_{z=-1}$, a positive mean upwelling requires $0 \geq m \geq 2$. In this parameterization, a larger |m| corresponds to a strong stratification near the surface (Fig. 3).

4.3. *The case of constant diffusivity*

The solution for the singular case of constant diffusivity can be obtained from the general solution in the limit $m \to 0$ because this particular case yields to a vanishing entropy variation (60), which can be verified directly. However, the same result is obtained by introducing the additional constraint of a fixed lateral flux of gravitational energy. In the limit of vanishing m, one finds the solution for constant diffusivity as

$$N^2(z) = N^2|_{z=-1}\, \mathrm{e}^{\frac{w_{s0}(z+1)^2}{2\kappa}} = N^2|_{z=0}\, \mathrm{e}^{-\frac{w_{s0}}{2\kappa}+\frac{w_{s0}(z+1)^2}{2\kappa}} \tag{66}$$

Assuming a vanishing buoyancy flux at the bottom is equivalent to the assumption $N|_{z=-1} = 0$. This leads to a singular solution, $N = 0$, $w_s \to \infty$. The vanishing heat flux condition at the ocean bottom can be relaxed through the assumption of a weak uniform diffusive upward mass flux from a bottom boundary layer such that the buoyancy frequency is $N^{\mathrm{dim}}_{\mathrm{bottom}}$ $5 \times 10^{-4}\mathrm{s}^{-1}$ for a diffusivity, $\kappa_v^{\mathrm{dim}} = 10^{-4}\ \mathrm{m}^2\mathrm{s}^{-1}$; moreover, the bottom vertical velocity is kept null. Now, the value of the mean upwelling at the surface (or at the base of the thermocline) can be estimated; using (66) leads to the following dimensional relation:

$$w_{s0}^{\mathrm{dim}} = \frac{2\kappa_{\mathrm{v}}^{\mathrm{dim}}}{H} \ln\left[\frac{\left(N^{\mathrm{dim}}\big|_{z=0}\right)^2}{\left(N^{\mathrm{dim}}\big|_{z=-1}\right)^2}\right]. \tag{67}$$

A typical value for the ratio of N between the bottom and the base of the thermocline is approximately 10. The corresponding value of the upwelling is, thus, $2 \times 10^{-7}\ \mathrm{m\ s}^{-1}$ for the above-mentioned values. The typical vertical length scale for such a profile is $\sqrt{\frac{H\kappa_{\mathrm{v}}^{\mathrm{dim}}}{w_{s0}^{\mathrm{dim}}}} \simeq 1.3 \times 10^3$ m.

This length scale can be treated as a measure of the thickness scale of the abyssal thermocline. Note that the vertical variation of the density field scales as $\kappa_v^{1/2}$. This scaling coincides with the internal thermocline scaling (Samelson, 1999 and references therein).

Finally, let us compute the energy and the entropy production as functions of bottom buoyancy flux and upwelling for a constant diffusivity:

$$E_{\mathrm{s}} = -\frac{N^2|_{z=-1}\left[2\sqrt{\frac{w_{s0}}{2\kappa}} - \sqrt{\pi}\,\mathrm{Erfi}\left(\sqrt{\frac{w_{s0}}{2\kappa}}\right)\right]}{8\left(\frac{w_{s0}}{2\kappa}\right)^{3/2}}, \tag{68a}$$

$$\sigma = \frac{N^2|_{z=-1}\sqrt{\pi}\,\mathrm{Erfi}\left(\sqrt{\frac{w_{s0}}{2\kappa}}\right)}{2\sqrt{\frac{w_{s0}}{2\kappa}}}$$

$$= -\frac{\left(\frac{2w_{s0}}{k}\right)E_{\mathrm{s}}\mathrm{Erfi}\left(\sqrt{\frac{w_{s0}}{2\kappa}}\right)}{-2\sqrt{\frac{w_{s0}}{2\kappa}} + \sqrt{\pi}\,\mathrm{Erfi}\left(\sqrt{\frac{w_{s0}}{2\kappa}}\right)}, \tag{68b}$$

where Erfi is the imaginary error function. These relations allow us to use any couple of parameters from $N^2|_{z=-1}$, w_{s0} or E_{s} to determine the ocean state unambiguously. For $N^2|_{z=-1} = 1$ in (68), the variation of energy and entropy production with respect to the parameter $w_{s0}/2\kappa$ is shown in Fig. 4.

Increasing the upwelling enhances entropy and concomitantly lowers the (negative) stratification energy. Besides, on condition to keep the stratification energy fixed and to vary upwelling, any increase of the diapycnal flow enhances the production of entropy.

Though, in principle, we should prove that the computed extrema are a maximum before concluding this section, it is not worth doing it, here, because the case under study is so constrained by conservation of mass and energy that it is devoid of free parameter to be varied. The entropy production as approximated by (59) cannot be varied from its value because any

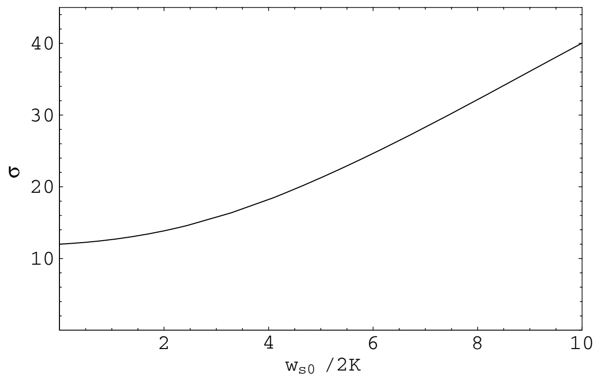

Fig. 5. Entropy production as a function of upwelling for a fixed energy ($E_s = -1$, a change in the value of E_s only changes the scale of the ordinate axis).

change in the density profile will violate the (potential) energy constraint (56) when either bottom stratification or surface upwelling is fixed. In the event our analysis would have provided us with a choice among various density profiles, we would have kept only the one producing the maximum entropy.

One may, now, wonder about what happens when one of the constraints, $E_s, N^2_{|z=-1}, w_{s0}$ is relaxed. The system will evolve till reaching a state where entropy production is the highest which implies maximizing upwelling. For example, considering the remaining components of the climate system needed to close the mass and energy balances while neglecting their contribution to the total mass, energy, and entropy will result in the system control by the abyssal ocean. If the bottom heat flux is fixed, the system will tend to greater entropy by indefinitely enhancing upwelling as shown in Fig. 5, which corresponds to a runaway situation. Actually, because of the finiteness of the system, the components of entropy and energy neglected in the above analysis will no longer be negligible; the inclusion of such additional processes (friction for example) should prevent the occurrence of such a runaway.

4.4. Horizontal circulation

The previous solutions hold in common a remarkable property: the vertical velocity profile is linear, and thus the divergence of horizontal velocity is constant. As a result, the Sverdrup balance (33d) leads to a barotropic flow such that

$$v_s = \tan\theta\, w_{s0} + \frac{T^w \delta\,(\phi - \phi_w)}{\cos\theta}, \tag{69}$$

where T^{w} indicates the transport or the western boundary current. Though our new solution seems to be the same as the classical solution for the homogeneous single-layer model by Stommel and Arons (1960a, 1990b), it strongly differs because it concerns a continuously stratified abyssal ocean. For the simple geometry of a pie-shaped model ocean (Pedlosky 1996, p. 389), the associated pressure is

$$p_s = p_{bc,s} + p_{bt,s} = -\int_0^z dz' \int_{z_s}^{z'} dz'' N^2(z'') + w_{s0}\,(\phi - \phi_e)\sin^2\theta + \sin\theta\, T^{\mathrm{w}}(\theta)\,[H\,(\phi - \phi_w) - 1]\,, \tag{70}$$

where z_s can be chosen to satisfy an additional constraint stating that the integrated mass anomaly is zero to satisfy (6). In this study we also assumed that the baroclinic anomaly of the reference pressure was zero at the upper surface of the model ocean. We also included in (69) and (70) a western boundary current as a delta distribution for the meridional velocity and thus as a Heaviside distribution for the barotropic pressure $p_{bt,s}$. The western boundary current intensity was obtained by using the sources spatial distribution (Stommel and Arons, 1960a, 1990b; Pedlosky, 1996) to integrate:

$$d_\theta T^{\mathrm{w}} = -2 w_{s0} \cos\theta. \tag{71}$$

However, the barotropic character of the circulation is directly related to the choice of the diffusivity parameterization. For instance, for a diffusivity coefficient depending on the unique variable N, one can easily verify that only a parameterization in the general form:

$$\kappa = aN^m + bN^{-2}, \quad (a, b) \in \Re^2, \tag{72}$$

yields a barotropic circulation. Incidentally, the constant diffusion case is part of this form. It is worth noting that if the parameter *a* (and *b*) was depth dependent, for example through the influence of the topography, the circulation would be different.

4.5. Energetics

In this section the mean density is used to satisfy (5); relation (6) is therefore valid and can be used to easily verify that the bottom pressure is 0 like the top baroclinic one. In addition to (6), the following relation holds:

$$\frac{d^2 p_{bc,s}}{dz^2} = -\frac{d\rho_s}{dz} = N^2 > 0. \tag{73}$$

As a result, the baroclinic pressure $p_{bc,s}$ is a concave function of z, and then the baroclinic pressure is always positive and can vanish only at the upper and lower boundaries.

At the first order in ε, eq. (18) can be rewritten as

$$0 = \int_{-1}^{0} \rho_s z S(z)\, dz + A \int_{-1}^{0} p_s \frac{dw_s}{dz}\, dz - w_{s0}\, p_{bt,s} + A \int_{-1}^{0} z \frac{d}{dz}\left(\kappa \frac{d\rho_s}{dz}\right) dz. \tag{74}$$

The proposed solutions (65) are such that the baroclinic contributions of the first two terms of (74) are equal. Indeed, by using the linear dependence of w_s, one can easily compute:

$$\int_{-1}^{0} \mathrm{d}z \rho_s z S(z) = A w_{s0} \int_{-1}^{0} \mathrm{d}z \rho_s z = A w_{s0} \left([-p_{bc,s} z]_{z=-1}^{z=0} + \int_{-1}^{0} \mathrm{d}z p_{bc,s} \right) = A w_{s0} \int_{-1}^{0} \mathrm{d}z\ p_{bc,s} = A \int_{-1}^{0} p_{bc,s} \frac{\mathrm{d}w_s}{\mathrm{d}z}, \qquad (75)$$

On the other hand, the barotropic contributions of the second and third terms cancel each other out. Thus, for the particular parameterization assumed here and subject to the constraint of stratified gravitational potential energy conservation, there is an equal partition of the energy provided by the mixing into the two sinks: the lateral fluxes and the pressure work.

5. Conclusion

We provided, here, a derivation of the abyssal stratification and circulation by using the principle of maximal information entropy production under the integral constraints of fixed global mass and energy. It is the reference state for an interior ocean that receives not only external mechanical energy for mixing, but also exchanges fluid and gravitational potential energy with the convective region. The latter prevents the unbounded growth of the gravitational potential energy of the system, as it could happen in a closed system (Winters et al., 1995) by providing a sink for this energy. The exchanges between the interior and the convective region is thus crucial for the existence of a steady state. Furthermore, we applied the maximum entropy production principle to the abyssal interior and thus de facto assumed that the convective region was slaved to the interior such as to provide the right amount of energy through the boundary fluxes. The maximum entropy production solution has maximum energy (here essentially potential energy) supply for a given (potential) energy and a given mass. Conversely, with a given (potential) energy supply and a given mass, the solution (potential) energy is at its minimum.

We derived solutions for different mixing parameterizations. In the case of a constant diffusivity, the vertical variation of the density field scales as $\kappa_v^{1/2}$, consistent with the scaling of the internal thermocline (Samelson, 1999). The derived stratification appears here to be intrinsically connected to the Stommel and Arons (1960a, 1960b) solution; although their solution is based on a single moving layer of constant density. In addition, a uniform upwelling, balanced by a prescribed source of deep water, is prescribed at the upper surface. The essential feature of the principle of maximal entropy production used here is that it simultaneously provides both the circulation and the mean stratification. Incidentally, the circulation obtained for the cases discussed in this study is barotropic. This barotropic circulation in the abyssal ocean is valid as long as diffusivity is a function of a negative power of N. Although such a negative power dependency has been discussed in previous publications, the situation in the oceans is much more complicated, and a comprehensive understanding of the connection between stratification and diffusivity remains a great challenge.

One should be aware that we only treated the component of the flow associated with the horizontally averaged stratification for a simple geometry. Departure from this simple stratification profile may bring in new features of the solutions induced by changes in the global constraint (energy, heat content, and volume); but, this part of a complete circulation is still to be determined.

Moreover, it should be noted that the results derived from the maximum entropy production principle are intimately related to the imposed constraints. For example, different stratification profiles were used to explore circulation configurations, but without discussing potential dynamical effects due to changes in the constraints related to the surface heat flux or the vertical velocity at the top of the abyss model ocean. It is likely that any change in these constraints would alternate substantially the circulation.

6. Acknowledgment

MBJ and RXH were supported by Ocean and Climate Change Institute, Woods Hole Oceanographic Institution.

References

Dewar, R. 2003. Information theory explanation of the fluctuation theorem, maximum entropy production and self-organized criticality in non-equilibrium stationary states. *J. Phys. A* **36**, 631.

Edwards, C. A. and Pedlosky, J. 1995. The influence of distributed sources and upwelling on the baroclinic structure. *J. Phys. Oceanogr.* **25**, 2259–2284.

Gargett, A. E. 1984. Vertical eddy diffusivity in the ocean interior. *J. Mar. Res.* **42**, 359–393.

Huang, R. X. 1993. A two-level model for the wind- and buoyancy-forced circulation. *J. Phys. Oceanogr.* **23**, 104–115.

Huang, R. X. 1999. Mixing and energetics of the thermohaline circulation. *J. Phys. Oceanogr.* **29**, 727–746.

Huang, R. X. 2004. Ocean, energy flow. In: *Encyclopedia of Energy* Volume 4 (ed. C. J. Cleveland). Elsevier, Oxford, 497–509.

Huang, R. X. and Wang, W. 2003. Gravitational potential energy sinks in the oceans. In: *Near-Boundary Processes and Their Parameterization*, Proceedings, Hawaii winter workshop, 239–247.

Jaynes, E. T. 1979. Where do we stand on maximum entropy? In: *The Maximum Entropy Formalism* (eds. R. D. Levine and M. Tribus). MIT Press, Cambridge, MA, p. 15.

Ledwell, J. R., Watson, A. J. and Law, C. S. 1993. Evidence for slow mixing across the pycnocline fron an open-ocean tracer-release experiment. *Nature* **364**, 701–703.

Munk, W. H. 1966. Abyssal recipes. *Deep-Sea Res.* **13**, 707–730.

Munk, W. H. and Wunsch, C. 1998. The Moon and mixing: abyssal recipes II. *Deep-Sea Res. I* **45**, 1977–2010.

Pedlosky, J. 1992. The baroclinic structure of the abyssal circulation. *J. Phys. Oceanogr.* **22**, 652–659.

Pedlosky, J. 1996. *Ocean Circulation Theory*. Springer-Verlag, Berlin, 460.

Polzin, K. L., Toole, J. M., Ledwell, J. R. and Scmitt, R. W. 1997. Spatial variability of turbulent mixing in the abyssal ocean. *Science* **276**, 93–96.

Sandström, J. W. 1908. Dynamicsche Versuche mit Meerwasser. *Annalen der Hydrographie und Maritimen Meteorologie* **36**, 6–23.

Samelson, R. M. 1999. Internal boundary layer scaling in "Two Layer" solutions of the thermocline equations. *J. Phys. Oceanogr.* **29**, 2099–2102.

Stigebrandt, A. 1979. Observational evidence for vertical diffusion driven by internal waves of tidal origin in the Olsofjord. *J. Phys. Oceanogr.* **9**, 435–441.

Stigebrandt, A. and Aure, J. 1989. On vertical mixing in the basin waters of fjords. *J. Phys. Oceanogr.* **19**, 917–926.

Stommel, H. and Arons, A. B. 1960a. On the abyssal circulation of the world ocean-I. Stationary planetary flow patterns on a sphere. *Deep-Sea Res.* **6**, 140–154.

Stommel, H. and Arons, A. B. 1960b. On the abyssal circulation of the world ocean-II. An idealized model of the circulation pattern and amplitude in oceanic basins. *Deep-Sea Res.* **6**, 217–233.

Winters, K. B., Lombard, P. N., Riley, J. J. and D'Asaro, E. A. 1995. Available potential energy and mixing in density-stratified fluids. *J. Fluid Mech.* **289**, 115–128.

Wunsch, C. and Ferrari, R. 2004. Vertical mixing, energy, and the general circulation of the oceans. *Ann. Rev. Fluid Mech.* **36**, 281–314.